Proc. Indian Acad. Sci. (Math. Sci.) (2019) 129:76
https://doi.org/10.1007/s12044-019-0504-x

Non-comaximal graph of ideals of a ring

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MS received 27 July 2018; accepted 8 January 2019

Abstract. Let R be a ring. The non-comaximal graph of R, denoted by $NC(R)$ is an undirected graph whose vertex set is the collection of all non-trivial (left) ideals of R and any two distinct vertices I and J are adjacent if and only if $I + J \neq R$. The concepts of connectedness, independent set, clique and traversability of $NC(R)$ are discussed.

Keywords. Non-comaximal graph; maximal ideal; independence set.

2010 Mathematics Subject Classification. 05C25.

1. Introduction

Now-a-days, the study of correspondence between algebraic structure and graph is an optimal research area. The zero divisor graph of a commutative ring is a milestone of this trend which was introduced in 1988 by Beck [5]. Some other graphs are introduced, which make a bridge between algebraic structure and graph. Sharma and Bhatwadekar [12] introduced the comaximal graph of a commutative ring with unity. Chakrabarty *et al.* [6] discussed the intersection graph of ideals of rings. The total graph of commutative ring was introduced by Anderson and Badawi in [3]. These graphs get more attention for extension.

For continuation of this sequel, we recollect some basic definitions and notations, which are already in literature. Throughout this discussion, all graphs are undirected. Let G be an undirected graph with the vertex set $V(G)$, unless otherwise mentioned. The degree of a vertex v in a graph G is the number of edges incident with v. A walk in G is an alternating sequence of vertices and edges, $v_0 x_1 v_1 \ldots x_n v_n$ in which each edge x_i is $v_{i-1}v_i$. A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a circuit is a closed walk with all points distinct (except the first and the last). The length of a circuit is nothing but the number of edges in the circuit. The length of the smallest circuit of G is called the girth of G, denoted by $\text{girth}(G)$. G is connected if there is a path between every two distinct vertices. A graph which is not connected is called a disconnected graph. A totally disconnected graph does not contain any edges. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path, we define $d(x, y) = \infty$. The diameter of G is the maximum distance among distances between all pair of vertices of G, denoted by $\text{diam}(G)$. If in the graph G any two vertices

Published online: 03 August 2019

are adjacent, it is called a complete graph. A complete subgraph of G is called a clique. A maximum clique of G is a clique with largest number of vertices and the number of vertices of a maximum clique is called the clique number of G. An independent set of G is a set of vertices of G which are not mutually adjacent. A maximum independent set of G is an independent set with largest number of vertices and the number of vertices of a maximum independent set is called the independence number of G, denoted by $\omega(G)$. An Eulerian circuit is a closed walk which contains all the edges of G exactly once. G with an Eulerian circuit is called an Eulerian graph. A Hamiltonian circuit is a spanning circuit in G. G is called Hamiltonian if it has a Hamiltonian circuit.

Any undefined terminologies are available in [1,2,4,7–11,13].

For a ring R, the non-comaximal graph is a graph whose vertex set is the collection of all non-trivial (left) ideals and any two vertices I, J are adjacent if and only if $I + J \neq R$. We denote the non-comximal graph of R by $NC(R)$ and follow this notation for the continuation of this article. In the next section, we discuss the concept of connectedness of $NC(R)$ of R. In that context, we find the diameter and girth of $NC(R)$. The insights of coloring and traversability are interpreted in section 3. We establish some results related with independence number and cliques. Finally, we conclude the section with some results of traversability and independent set. Most of the results are established in the ring Z_n, the ring of integer modulo n. From here onwards, unless otherwise specified, R is a ring with unity.

2. Connectedness of $NC(R)$

This section contains some results of connectedness of $NC(R)$ and $NC(Z_n)$.

Theorem 2.1. *$NC(R)$ is complete if and only if ideals of R form a chain.*

Proof. If the ideals of R form a chain, then it is easy to check that any two vertices of $NC(R)$ are adjacent. For the converse part, we consider $NC(R)$ is a complete graph. On the contrary, assume that there exists two ideals I and J of R such that either one of them is not contained in the other. Now consider a maximal ideal M which contains I and this yields that J and M are not adjacent. Hence the theorem. □

Theorem 2.2. *$NC(Z_n)$ is disconnected if and only if $n = p_1 p_2$, where p_1 and p_2 are distinct primes.*

Proof. If $n = p_1 p_2$, then $(p_1) + (p_2) = Z_n$ and so $NC(Z_n)$ is disconnected. For the opposite direction, assume the prime factorization of $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$. If $r > 2$, then the vertices (p_1) and $(p_1 p_2)$ are adjacent. Again if $k_i \geq 2$ for any $i = 1, 2, \dots, r$, then the vertices (p_i) and (p_i^2) are adjacent. Hence $NC(Z_n)$ is disconnected if and only if $n = p_1 p_2$. □

In fact, $NC(Z_n)$ is totally disconnected if and only if $n = p_1 p_2$, where p_1 and p_2 are distinct primes. From this, we see that $NC(Z_n)$ is not bicolorable if and only if $n = p_1 p_2$. The next theorem establishes the necessary and sufficient condition of totally disconnectedness for an arbitrary ring with unity.

Theorem 2.3. *$NC(R)$ is totally disconnected if and only if every non-trivial ideal of R is maximal as well as minimal.*

Proof. Let every non-trivial ideal of R be maximal as well as minimal. Consider two distinct non-trivial ideals I and J. Then $I + J = R$, and as I and J are arbitrary, there is no edge between any pair of distinct ideals. Thus $NC(R)$ is totally disconnected. For the other direction, suppose that I and J are two non-trivial ideals of R. Then totally disconnectedness of $NC(R)$ gives that $I + J = R$. If $I \cap J \neq 0$, then $I - I \cap J - J$ is a path. This contradiction asserts that $I \cap J = 0$. If both I and J are not maximal, then we have two maximal ideals M_1 and M_2 with $I \subseteq M_1$ and $J \subseteq M_2$, respectively. If $M_1 = M_2$, then we obtain a path $I - M_1 - J$ and so M_1 and M_2 are distinct. Without loss of generality, we assume that I is not maximal. Let $x \in M_1$ and $x \notin I$. Since $M_1 + M_2 = R$, there exist $i \in I$ and $j(\neq 0) \in J$ with $x = i + j$. This gives that $M_1 \cap M_2 \neq 0$ as $j \in M_1 \cap M_2$. This concludes that $I - M_1 \cap M_2 - J$ is a path, a contradiction. Hence the theorem. □

Next, we find the girth of $NC(Z_n)$. Then we generalize the result for $NC(R)$.

Theorem 2.4. *If $NC(Z_n)$ contains a circuit, then* $\mathrm{girth}(NC(Z_n)) = 3$.

Proof. Observe that n is not equal to p^2, p^3 or $p_1 p_2$. If $n = p_r, r > 3$, then $NC(Z_n)$ is a complete graph. In general, if $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of n, then the vertices (p_i), (p_i^2) and $(p_i p_j)$ form a circuit whenever $k_i > 1, i = 1, 2, \ldots, r$. Hence the result. □

Theorem 2.5. *If $NC(R)$ contains a circuit, then* $\mathrm{girth}(NC(R)) = 3$.

Proof. Notice that it is not the case here that every non-trivial ideal of R is maximal as well as minimal. If $I + J \neq R$ for ideals I and J, then we obtain a circuit $I - J - I + J - I$. Again, if $I + J = R$, then there are maximal ideals M_1, M_2 with $I \subseteq M_1$ and $J \subseteq M_2$, respectively. Like the proof of previous theorem, we can get the circuit $I - M_1 - M_1 \cap M_2 - I$. Hence $\mathrm{girth}(NC(R)) = 3$. □

Theorem 2.6. *For Z_n,* $\mathrm{diam}(NC(Z_n)) = 1, 2$ *or* ∞.

Proof. If $n = p_1 p_2$, then $\mathrm{diam}(NC(Z_n)) = \infty$. Again, if $n = p^r, r > 2$, then $\mathrm{diam}(NC(Z_n)) = 1$. In general, consider the case when the prime factorization of n is $p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. If (l) and (m) are two vertices such that l and m have no common prime divisor, then (l) and (m) are not adjacent. In this case, we can find a prime divisor of m and multiple of that prime divisor with l gives a vertex (q), where q is the multiple of l with that prime divisor. Thus, we get a path $(l) - (q) - (m)$ and so $\mathrm{diam}(NC(Z_n)) = 2$. This completes the proof. □

Remark 2.7. Recollect that R is local means R contains exactly one maximal ideal.

Theorem 2.8. *For R,* $\mathrm{diam}(NC(R)) = 1, 2$ *or* ∞.

Proof. If every non-trivial of R is maximal as well as minimal, then $\text{diam}(NC(R)) = \infty$. Again, if R is a local ring, then $\text{diam}(NC(R)) = 1$. Lastly, if every non-trivial ideal of R is not maximal as well as minimal, then for vertices I and J, we have maximal ideals M_1 and M_2 with $I \subseteq M_1$ and $J \subseteq M_2$, respectively. As earlier in Theorem 2.3, we can show that $I - M_1 \cap M_2 - J$ is a path. Thus $\text{diam}(NC(R)) = 2$. Hence the theorem. □

3. Independence set, clique and traversability of $NC(R)$

In this section, we discuss the concepts of independence set, clique and traversability of $NC(R)$. We look at these insights in $NC(Z_n)$ also. It is necessary to mention here that in the ring Z_n, $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime factorization n for the following sequel.

Remark 3.1. First, we find the independence number of $NC(Z_n)$. Notice that an independent set contains those ideals which are generated by powers of distinct primes. Take the independent set $I = \{(p_1^{j_1}), (p_2^{j_2}), \dots, (p_r^{j_r})\}$, where $1 \leq j_l \leq k_l, l = 1, 2, \dots, r$. No two vertices of I are adjacent. Therefore, I is a maximum independent set. Thus independence number of $NC(Z_n)$ is r. □

Theorem 3.2. *The number of maximum independent set of $NC(Z_n)$ is $k_1 k_2 \dots k_r$.*

Proof. Let $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ be the prime factorization of n. By Remark 3.1, the maximum independent set contains r members and all these members are generated by powers of distinct primes. There are $k_1 k_2 \dots k_r$ number of sets. Hence the number of maximum independent sets is $k_1 k_2 \dots k_r$. □

Theorem 3.3. *The independence number of $NC(R)$ is $|\max(R)|$, where $\max(R)$ is the set of all maximal ideals in R.*

Proof. First, note that if an ideal I belongs to an independent set, then the ideal, which is contained in the corresponding maximal ideal of I is not a member of that independent set. It is just a routine work to see that no two members of $\max(R)$ are adjacent. If we want to add ideal I in $\max(R)$, then there must exist a maximal ideal M for I and so I and M are adjacent. This violates the independence character of $\max(R)$. Thus $\max(R)$ is a maximum independent set. Therefore, the independence number of $NC(R)$ is $|\max(R)|$. Hence the theorem. □

Theorem 3.4. *The clique number of $NC(Z_n)$ is $k_t(k_1 + 1)(k_2 + 1) \dots (k_{t-1} + 1)(k_{t+1} + 1) \dots (k_r + 1) - 1$, where k_t is the maximum value of $k_i, i = 1, 2, \dots, r$.*

Proof. The vertex set of $NC(Z_n)$ has $T_1 = (k_1 + 1)(k_2 + 1) \dots (k_r + 1) - 2$ elements. Consider the element $(p_i^{j_i})$, where $1 \leq j_i \leq k_i, 1 \leq i \leq r$. Then $(p_i^{j_i})$ is adjacent to all vertices which are multiples of p_i. There are $T_2 = (k_1 + 1)(k_2 + 1) \dots (k_{i-1} + 1)(k_{i+1} + 1) \dots (k_r + 1) - 1$ vertices which are not multiples of p_i. If C is the set of vertices which are generated by multiples of p_i, then number elements in C is $T_1 - T_2 = k_i(k_1 + 1)(k_2 + 1) \dots (k_{i-1} + 1)(k_{i+1} + 1) \dots (k_r + 1) - 1$. The number of elements in C will be maximum for the maximum value of $k_i, i = 1, 2, \dots, r$. Thus the clique number

$NC(Z_n)$ is $k_t(k_1+1)(k_2+1)\dots(k_{t-1}+1)(k_{t+1}+1)\dots(k_r+1)-1$, where k_t is the maximum value of k_i, $i=1,2,\dots,r$. The proof is complete. □

Remark 3.5. Now we see the process of counting the total number of cliques of $NC(Z_n)$, where $n = p_1^{k_1}p_2^{k_2}\dots p_r^{k_r}$ is the prime factorization. First, we determine the number of ideals which are generated by multiple of p_1. There are k_1 number of ideals of the form $(p_1^{j_1})$, where $0 < j_1 < k_1$. There are $k_1(k_2+k_3+\dots+k_r)$ ideals of the form $(p_1^{j_1}p_m^{j_m})$, where $0 \le j_1 \le k_1$ and $1 \le j_m \le k_m$, $m = 2,3,\dots,r$. Following this way, we obtain the number of ideals generated by multiples of p_1 is $T_1 = k_1(k_2+k_3+\dots+k_r)+k_1k_2(k_3+k_4+\dots+k_r)+k_1k_3(k_4+k_5+\dots+k_r)+\dots+k_1k_{r-1}k_r+k_1k_2k-3\dots k_r-1$. Thus cliques with $2,3,\dots,T_1$ members from T_1 is given by $\binom{T_1}{2}, \binom{T_1}{3}, \dots, \binom{T_1}{T_1}$, respectively. In the same way, the number of ideals generated by p_2, but not by p_1 is given by $T_2 = k_2 + k_2k_3(k_4+k_5+\dots+k_r)+\dots+k_2k_3k_4\cdots k_r$. The cliques from this T_2 elements with $2,3,\dots,T_2$ elements are given by $\binom{T_2}{2}, \binom{T_2}{3}, \dots, \binom{T_2}{T_2}$, respectively. Proceeding these steps r times, we can count the total number of cliques in $NC(Z_n)$. □

In the next theorem, we obtain the clique number in $NC(R)$.

Theorem 3.6. *The clique number of $NC(R)$ is the cardinality of the set containing all the ideals, except zero, which are contained in the maximal ideal M, where M will contain the maximum number of ideals among all maximal ideals of R.*

Proof. Let $\mathfrak{M}$ be the set of all ideals which are contained in M, where M is the maximal ideal containing maximum number of ideals among all maximal ideals of R. Then any two members of $\mathfrak{M}$ are adjacent. If we add any other ideal in $\mathfrak{M}$, then $\mathfrak{M}$ is no longer a set of mutually adjacent vertices. This asserts that $\mathfrak{M}$ is a maximum independent set. Hence the clique number of $NC(R)$ is the cardinality of $\mathfrak{M}$. This completes the proof. □

Next, we establish some results related with the concepts of traversability.

Theorem 3.7. *$NC(Z_n)$ is Eulerian, if each k_i is even*; $i = 1,2,\dots,r$.

Proof. The vertex set of $NC(Z_n)$ contains $T_1 = (k_1+1)(k_2+1)\cdots(k_r+1)-2$ elements. Let each k_i be even, $i = 1,2,\dots,r$. Then the vertex set of $NC(Z_n)$ contains $T_1 = (k_1+1)(k_2+1)\cdots(k_r+1)-2$ elements which is an odd number. Now consider a vertex $I = (p_1^{j_1}p_2^{j_2}\dots p_t^{k_t})$, where $t \ge 1$, $0 \le j_i \le k_i$. Thus I is adjacent to all vertices which are generated by multiples of p_i, where $i = 1,2,\dots,t$. Therefore, there are even number of $T_2 = (k_t+1)(k_t+2)\cdots(k_r+1)-1$ vertices which are not adjacent to I. Considering this, we conclude that I is an even degree vertices of degree $T_1 - T_2 - 1$. Hence $NC(Z_n)$ is an Eulerian graph. The proof is complete. □

Remark 3.8. Observe that the converse of the above theorem is not true. This can be obtained by looking in the Eulerian graph $NC(Z_n)$ for $n = pqr$.

Theorem 3.9. *If R is finite and every maximal ideal of R contains even number of non-trivial ideals, then $NC(R)$ is Eulerian.*

Proof. Let M be a maximal ideal of R. If $\mathfrak{M}$ be the collection of non-trivial ideals which are contained in M, then $|\mathfrak{M}|$ is an even number. Then any two members in $\mathfrak{M}$ are adjacent. So each member in $\mathfrak{M}$ is of degree $|\mathfrak{M}|$. Hence $NC(R)$ is Eulerian. □

Theorem 3.10. *$NC(Z_n)$ is Hamiltonian if and only if $n \neq p^2, p^3, p_1p_2, p_1^2p_2$ or $p_1p_2^2$.*

Proof. One part is evident. For the converse part, take $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$. If we consider the collection of vertices which are generated by multiples of p_1 except (p_1p_2) and (p_1p_r), then we can get a circuit C_1 which contains all the vertices of the collection. Now leaving the vertices of C_1, if we consider the collection of vertices which are generated by multiples of p_2, except (p_2p_3), we obtain a circuit C_2 which contains all the vertices of the collection. Following this method, we find a circuit C_r, where the vertices in C_r are multiples of p_r but not multiples of any of $p_1, p_2, \dots, p_{r-1}$. Then $(p_1) - C_1 - (p_1p_2) - C_2 - (p_2p_3) - \cdots - C_r - (p_1p_r) - (p_1)$ is a Hamiltonian circuit in $NC(Z_n)$. Hence the theorem. □

In the last part of this section, we establish some results related with independence number of $NC(R)$, where R is a commutative ring with unity.

Theorem 3.11. *If R has infinite number of idempotents, then $NC(R)$ contains an infinite independent set.*

Proof. Let x be a non-trivial idempotent of R. Then $R = Rx \times R(1-x)$ can be written as the direct product of the rings Rx and $R(1-x)$. Using the given condition, we either have Rx or $R(1-x)$ containing infinitely many idempotents. Thus there exists a non-trivial idempotent y_1 with Ry_1 containing infinitely many idempotents. Since y_1 is a non-zero idempotent, we can consider Ry_1 as a ring with unity y_1. In the same way, we get a non-zero idempotent $y_2 (\neq y_1)$ with Ry_2 containing infinitely many idempotents. Proceeding in this direction, we have an infinite sequence $y_1, y_2, \dots, y_n, \dots$ of non-trivial idempotents such that $Ry_{i+1} \subsetneqq Ry_i$ for every $i = 1, 2, \dots$. Take $y_0 = 1$ and $x_{i+1} = y_{i+1} + 1 - y_i$ for $i \geq 0$. Then for every positive integer n, x_n is a non-trivial idempotent such that $y_n = \prod_{j=1}^{n} x_j$ and $Ry_n + Rx_{n+1} = R$. Hence $\{Rx_n : n \geq 1\}$ is an independent set of $NC(R)$. □

Theorem 3.12. *If $NC(R)$ does not contain any infinite independent set, then R is a finite ring.*

Proof. We assume that R is a reduced ring with the Jacobson radical $J(R)$ of R is zero. We can consider this assumption because of the following observation. Since $NC(R)$ does not contain any infinite independent set, the set of units G of R is a finite multiplicative group. Thus $J(R)$ is a finite ideal, as the map $x \to 1 + x$ is an injective mapping of $J(R)$ into G. Again, by assumption, $R/J(R)$ does not contain any infinite independent set. So, to prove the theorem, we replace R by $R/J(R)$. Since the ideals generated by prime numbers of Z form an infinite independent set, therefore, Z cannot be a prime subring of R. Thus R has positive characteristics. This helps to express R as a product $R = R_1 \times R_2 \times \cdots \times R_s$, where characteristics of each R_i is a prime number for $i = 1, 2, \dots, s$. Immediately, we can assume that characteristics of R is a prime number p, say. This will provide that $Z/(p)$ is a subring of R. Since the collection of all ideals generated by monic irreducible polynomials is an infinite independent set of the polynomial ring $Z/(p)[x]$, every element of R is algebraic over $Z/(p)$. In this case, the Krull dimension of R is zero. But then, every principal ideal of R is an idempotent ideal as R is reduced. By Theorem 3.11, R has finitely many idempotents. From this, it is obtainable that R is a direct product of a finite number of indecomposable zero-dimensional reduced ring, i.e. fields. Clearly, R is a finite direct product of finite fields, as G is a finite group. Therefore, R is a finite ring. Hence the theorem. □

Theorem 3.13. *The ring R is finite if and only if $\omega(NC(R))$ is finite and $\omega(NC(R)) = t + l$, where t is the number of maximal ideals of R and l denotes the number of units of R.*

Proof. Clearly, if R is finite then $\omega(NC(R))$ is finite. In the other direction, if $\omega(NC(R))$ is finite, then every independent set of $NC(R)$ is finite. Hence, by the above theorem, R is a finite ring.

To establish the second part, we consider $G = \{u_1, u_2, \ldots, u_l\}$ as the collection of all units of R and $\{M_1, M_2, \ldots, M_t\}$ is the collection of all maximal ideals of R. Now using the Chinese Remainder theorem, R contains an element $a_i \in M_i$ with $a_i - 1 \in M_j, \forall i \neq j, i = 1, 2, \ldots, t$. But this gives that $S = \{(u_1), (u_2), \ldots, (u_l), (a_1), (a_2), \ldots, (a_t)\}$ is an independent set of $NC(R)$ and thus $\omega(NC(R)) \geq l + t$. If we can show that $\omega(NC(R)) \leq l + t$, then this will prove the required result. For this, we define $T_1 = M_1$ and $T_k = M_k - \bigcup_{i=1}^{k-1} M_i, 2 \leq k \leq t$. Evidently, $G \cup T_1 \cup T_2 \cup \cdots \cup T_t$ is a partition of R. Now consider the map $f : R \longrightarrow \{1, 2, \ldots, t + l\}$, defined by $f(u_i) = i$ for $1 \leq i \leq l$ and $f(T_j) = l + j$ for $1 \leq j \leq t$. It is clear that if x and y are two distinct elements of R with $(x) + (y) = R$, then $f(x)$ and $f(y)$ are distinct. Thus we have $\{(x_i) : x_i \in R\}$ is a maximum independent set of $NC(R)$ and so $\omega(NC(R)) \leq l + t$. This completes the proof. □

COROLLARY 3.14

If R is a finite local ring having residue field consisting of p^n elements, then p is a prime. Then $\omega(NC(R)) = p^{nm} - p^{n(m-1)} + 1$, where m is the length of R.

Example 3.15. Let $R = Z/(2) \times \cdots \times Z/(2)$. It is easy to check that R has evidently $n - 1$ maximal ideals and unity is the only unit element of R. Therefore, by Theorem 3.13, we conclude that $\omega(NC(R)) = n$.

Acknowledgements

This paper is the outcome of INSPIRE Summer Research Project 2017, Department of Science and Technology, Government of India of the first author (INSPIRE Registration No. 201300036277, Application Reference No. DST/INSPIRE/02/2013/026713).

References

[1] Atiyah M F and Macdonald I G, Introduction to commutative algebra (1969) (London: Addison-Wesley)

[2] Anderson F W and Fuller K R, Rings and Categories of Modules, edited by P R Halmos (1978)

[3] Anderson D F and Badawi A, The total graph of a commutative ring, *J. Algebra* **320** (2008) 2706–2719
[4] Balkrishnan R and Ranganathan K, A Text Book of Graph Theory, reprint (2008) (New York: Springer-Verlag)
[5] Beck I, Coloring of commutative rings, *J. Algebra.* **116** (1988) 208–226
[6] Chakrabarty I, Ghosh S, Mukherjee T K and Sen M K, Intersection graphs of ideals of rings, *Discrete Math.* **309(17)** (2009) 5381–5392
[7] Goodearl K R, Ring Theory (1976) (Marcel Dekker)
[8] Harary F, Graph Theory (1969) (Reading, Mass.: Addison-Wesley Publishing Company Inc.)
[9] Kasch F, Modules and Rings (1982) (London: Academic Press Inc. Ltd)
[10] Kaplansky I, Commutative Rings, revised edition (1974) (Chicago: University of Chicago Press)
[11] Lambeck J, Lectures on Rings and Modules (1966) (Waltham, Toronto, London: Blaisdell Publishing Company)
[12] Sharma P K and Bhatwadekar S M, A note on graphical representation of rings, *J. Algebra* **176** (1995) 124–127
[13] Wisbauer R, Foundations of Module and Ring Theory, (1991) (Reading: Gordon and Breach Science Publishers)

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