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ORIGINAL RESEARCH

Solvability of generalized fractional order integral equations via measures of noncompactness

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Abstract

In this article, we work on the existence of solution of generalized fractional integral equations of two variables. To achieve our main objective, we establish a new fixed point theorem using measure of noncompactness and a new contraction operator which generalized the Darbo's fixed point theorem (DFPT). Also we obtain the corresponding coupled fixed point theorem. Finally we apply this generalized DFPT on the generalized fractional integral equations of two variables and illustrate our findings with the help of an example.

Keywords Measure of noncompactness (MNC) · Darbo's fixed point theorem (DFPT) · Functional integral equations

Mathematics Subject Classification 35K90 · 47H10

Introduction

Fractional calculus is the study of the derivatives as well as integrals of arbitrary order using Gamma function. A fractional derivative in applied mathematics and mathematical analysis is a derivative of any noninteger order, real or complex. The first existence is in a letter written by G.W. Leibniz in sixteenth century ending to Antoine de l'Hopital [20]. In one of N. H. Abel's early papers [1], fractional calculus was adopted, where those elements can be considered: the definition of integration and differentiation of fractional order, the strictly inverse connection among them, the perception that differentiation and integration of fractional order can be perceived as being in the same generalized operation, and indeed the coherent form for ambiguous real order differentiation and integration. Over the nineteenth and early twentieth centuries, the theory and applications of fractional calculus developed greatly, and countless contributors have provided interpretations for fractional derivatives and integrals. The Erdélyi–Kober fractional integrals are used in many branches of mathematics like porous media, viscoelasticity and electrochemistry, etc. (see [9, 18]). Fixed point theory and measure of noncompactness have many applications in solving different types of integral equations which overcome the different real-life situations see for instance [4, 5, 12, 24, 25, 27–30]. Due to the importance of integral

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equations of fractional order, it has become essential to study such type of equations.

Darbo's fixed point theorem and its generalizations which use the concept of MNC have been applied by many authors [2, 5, 11, 15, 16, 21–24] to study integral as well as differential equations. With the help of different type contraction of operators, Darbo's fixed point theorem has been generalized by different researchers in the recent past, see [11, 15, 23, 26]. Işik et. al. [19] have extended Darbo's fixed point theorem via weak JS-contractions in a Banach spaces, also derived couple fixed point theorem and applied it to study the existence of solutions for a system of integral equations. So motivated by these works, we have generalized the DFPT using a new contraction operator which is defined with the help of a function used in [17] and apply it on a generalized fractional integral equation of two variables to check the solvability.

Preliminaries

Let $(\mathfrak{E}, \|\,.\,\|)$ be a real Banach space. Suppose that $B(\theta, r) = \{x \in \mathfrak{E} : \| x - \theta \| \leq r\}$. If $\mathfrak{X}$ be a nonempty subset of $\mathfrak{E}$, then by $\bar{\mathfrak{X}}$ and Conv$\mathfrak{X}$ we denote the closure and convex closure of $\mathfrak{X}$, and let $\mathfrak{M}_{\mathfrak{E}}$ be the family of all nonempty and bounded subsets of $\mathfrak{E}$ and $\mathfrak{N}_{\mathfrak{E}}$ be its subfamily consisting of all relatively compact sets.

Definition 2.1 [6] A function $\vartheta : \mathfrak{M}_{\mathfrak{E}} \to [0, \infty)$ is said to be a MNC in $\mathfrak{E}$ if

(i) for all $\mathfrak{U} \in \mathfrak{M}_{\mathfrak{E}}$, we have $\vartheta(\mathfrak{U}) = 0$ which gives $\mathfrak{U}$ is relatively compact.
(ii) $\ker \vartheta = \left\{\mathfrak{U} \in \mathfrak{M}_{\mathfrak{E}} : \vartheta(\mathfrak{U}) = 0\right\} \neq \phi$ and $\ker \vartheta \subset \mathfrak{N}_{\mathfrak{E}}$.
(iii) $\mathfrak{U} \subseteq \mathfrak{U}_1 \implies \vartheta(\mathfrak{U}) \leq \vartheta\left(\mathfrak{U}_1\right)$.
(iv) $\vartheta\left(\bar{\mathfrak{U}}\right) = \vartheta(\mathfrak{U})$.
(v) $\vartheta(Conv\mathfrak{U}) = \vartheta(\mathfrak{U})$.
(vi) $\vartheta\left(\delta\mathfrak{U} + (1-\delta)\mathfrak{U}_1\right) \leq \delta\vartheta(\mathfrak{U}) + (1-\delta)\vartheta\left(\mathfrak{U}_1\right)$ for $\delta \in [0, 1]$.
(vii) if $\mathfrak{U}_j \in \mathfrak{M}_{\mathfrak{E}}$, $\mathfrak{U}_j = \bar{\mathfrak{U}}_j$, $\mathfrak{U}_{j+1} \subset \mathfrak{U}_j$ for $j = 1, 2, 3, \ldots$ and $\lim_{j\to\infty} \vartheta\left(\mathfrak{U}_j\right) = 0$ then $\bigcap_{j=1}^{\infty} \mathfrak{U}_j \neq \phi$.

The family $ker\vartheta$ is said to be the *kernel of measure* ϑ. The set $\mathfrak{U}_\infty = \bigcap_{j=1}^{\infty} \mathfrak{U}_j \in ker\vartheta$. Since $\vartheta(\mathfrak{U}_\infty) \leq \vartheta(\mathfrak{U}_j)$ for any j, we infer that $\vartheta(\mathfrak{U}_\infty) = 0$.

Theorem 2.2 [3, *Schauder*] *Let* $\mathfrak{U}$ *be a nonempty bounded closed and convex subset of a Banach space* $\bar{\mathfrak{E}}$. *Then every compact continuous map* $\mathfrak{G} : \mathfrak{U} \to \mathfrak{U}$ *admits at least one fixed point.*

The following theorem is the generalization of the above theorem (Schauder fixed point theorem (SFPT)) which is known as Darbo's fixed point theorem (DFPT).

Theorem 2.3 [14, *Darbo*] *Let* $\mathfrak{U}$ *be a nonempty, bounded, closed and convex subset of a Banach space* $\bar{\mathfrak{E}}$. *Let* $\mathfrak{S} : \mathfrak{U} \to \mathfrak{U}$ *be a continuous mapping and there is a constant* $\kappa \in [0, 1)$ *such that*

$$\vartheta(\mathfrak{S}\mathfrak{M}) \leq \kappa\vartheta(\mathfrak{M}), \ \mathfrak{M} \subseteq \mathfrak{U}.$$

Then $\mathfrak{S}$ *possesses a fixed point.*

To establish the extension of DFPT, we recall following related concepts

Definition 2.4 [17] Let $\mathcal{F}$ be the collection of all functions $\Delta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

(1) $\max\{\iota, \varpi\} \leq \Delta(\iota, \varpi)$ for $\iota, \varpi \geq 0$.
(2) Δ is continuous and nondecreasing.
(3) $\Delta(\iota + \varpi, \iota_1, \varpi_1) \leq \Delta(\iota, \iota_1) + \Delta(\varpi, \varpi_1)$.

For example, $\Delta(\iota, \varpi) = \iota + \varpi$.

Definition 2.5 [26] Let Γ denote the set of all functions $\gamma : \mathbb{R}_+ \to [1, \infty)$ such that

(1) γ is a continuous and strictly increasing function.
(2) for each sequence $\left\{\kappa_j\right\} \subseteq (0, \infty)$, $\lim_{j\to\infty} \gamma\left(\kappa_j\right) = 1$ if and only if $\lim_{j\to\infty} \kappa_j = 0$.

For example, $\gamma(\kappa) = e^{\kappa}$ belongs to Γ.

Definition 2.6 Let Ξ be the set of all functions $\xi : [1, \infty) \to \mathbb{R}_+$ satisfying

(1) ξ is continuous.
(2) $\xi(1) = 0$.
(3) for each sequence $\left\{\kappa_j\right\} \subseteq (1, \infty)$, $\lim_{j\to\infty} \xi\left(\kappa_j\right) = 0$ if and only if $\lim_{j\to\infty} \kappa_j = 1$.

For example

(1) $\xi_1(l) = l - l^{\frac{1}{n}}$, $n \geq 1$ belongs to Ξ,
(2) $\xi_2(l) = e^{l-1} - 1$ belongs to Ξ,
(3) $\xi_3(l) = \ln(l)$ belongs to Ξ.

Definition 2.7 [8] A mapping $S : [0, \infty) \to \mathcal{L}(X)$ is said to be a strongly continuous semigroup on X if the following conditions hold:

(i) $S(0) = I$, $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$
(ii) for all $x \in X$, $S(.)x$ is continuous on $[0, \infty)$, where X denotes a complex Banach space and $\mathcal{L}(X)$ denotes the Banach algebra of all linear continuous mappings.

Main result

Theorem 3.1 *Let $\mathfrak{B}$ be a nonempty bounded, closed and convex subset of a Banach space $\mathfrak{E}$. Also, let $\mathfrak{W} : \mathfrak{B} \to \mathfrak{B}$ be a continuous mapping such that*

$$\gamma\left[\Delta(\vartheta(\mathfrak{W}\mathcal{A}), \alpha(\vartheta(\mathfrak{W}\mathcal{A})))\right] \leq \gamma[\Delta(\vartheta(\mathcal{A}), \alpha(\vartheta(\mathcal{A})))] - \xi\{\gamma[\Delta(\vartheta(\mathcal{A}), \alpha(\vartheta(\mathcal{A})))]\} \tag{3.1}$$

for all $\mathcal{A} \subseteq \mathfrak{B}$, $\gamma \in \Gamma$, $\xi \in \Xi$, $\Delta \in \mathcal{F}$ where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous mapping, and ϑ is an arbitrary MNC. Then $\mathfrak{W}$ possesses at least one fixed point in $\mathfrak{B}$.

Proof Let $(\mathfrak{B}_j)$ with $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{B}_{j+1} = Conv(\mathfrak{W}\mathfrak{B}_j)$ for all $n \geq 0$. Also, $\mathfrak{W}\mathfrak{B}_0 = \mathfrak{W}\mathfrak{B} \subseteq \mathfrak{B} = \mathfrak{B}_0$, $\mathfrak{B}_1 = Conv(\mathfrak{W}\mathfrak{B}_0) \subseteq \mathfrak{B} = \mathfrak{B}_0$. Continuing in the similar manner gives $\mathfrak{B}_0 \supseteq \mathfrak{B}_1 \supseteq \mathfrak{B}_2 \supseteq \ldots \supseteq \mathfrak{B}_j \supseteq \mathfrak{B}_{j+1} \supseteq \ldots$ .If there exists $k \in \mathbb{N}$ satisfying $\vartheta(\mathfrak{B}_k) = 0$, then $\mathfrak{B}_k$ is a compact set. By SFPT, $\mathfrak{W}$ has a fixed point.

Let $\vartheta(\mathfrak{B}_j) > 0$ for some $j \in \mathbb{N}$. Clearly, the sequence $\{\vartheta(\mathfrak{B}_j)\}$ is a nonnegative, decreasing and bounded below sequence. So, the sequence is convergent and let $\lim_{j\to\infty} \vartheta(\mathfrak{B}_j) = b \geq 0$.

Also, $\vartheta(\mathfrak{B}_{j+1}) = \vartheta(Conv(\mathfrak{W}\mathfrak{B}_j)) = \vartheta(\mathfrak{W}\mathfrak{B}_j)$ and by (3.1) we have

$$\begin{aligned}&\gamma\left[\Delta\left(\vartheta\left(\mathfrak{B}_{j+1}\right), \alpha\left(\vartheta\left(\mathfrak{B}_{j+1}\right)\right)\right)\right]\\ &= \gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\mathfrak{B}_j\right), \alpha\left(\vartheta\left(\mathfrak{W}\mathfrak{B}_j\right)\right)\right)\right]\\ &\leq \gamma\left[\Delta\left(\vartheta\left(\mathfrak{B}_j\right), \alpha\left(\vartheta\left(\mathfrak{B}_j\right)\right)\right)\right] - \xi\left\{\gamma\left[\Delta\left(\vartheta\left(\mathfrak{B}_j\right), \alpha\left(\vartheta\left(\mathfrak{B}_j\right)\right)\right)\right]\right\}.\end{aligned}$$

If it is possible, assume that $b > 0$. As $j \to \infty$, we have

$$\gamma[\Delta(b, \alpha(b))] \leq \gamma[\Delta(b, \alpha(b))] - \xi\{\gamma[\Delta(b, \alpha(b))]\},$$

i.e., $\xi\{\gamma[\Delta(b, \alpha(b))]\} \leq 0$.

Hence, $\xi\{\gamma[\Delta(b, \alpha(b))]\} = 0$ which gives $\gamma[\Delta(b, \alpha(b))] = 1$. Consequently, $\Delta(b, \alpha(b)) = 0$ which gives $\lim_{j\to\infty} \vartheta(\mathfrak{B}_j) = b = 0$. Since $\mathfrak{B}_j \supseteq \mathfrak{B}_{j+1}$, by Definition 2.1, we get $\mathfrak{B}_\infty = \bigcap_{j=1}^{\infty} \mathfrak{B}_j$ is a nonempty, closed and convex subset of $\mathfrak{B}$ and $\mathfrak{B}_\infty$ is $\mathfrak{W}$ invariant. Thus, Theorem 2.2 implies that $\mathfrak{W}$ admits a fixed point in $\mathfrak{B}_\infty \subseteq \mathfrak{B}$. □

Remark 3.2 We have generalized Darbo's fixed point theorem by using a new contraction operator which involves MNC to study operators whose properties can be characterized as being intermediate between those of contraction and compact mapping. The main advantage of this generalization using MNC is that the compactness of domain of the operator which is essential in Shauder's theorem has been relaxed.

Theorem 3.3 *Let $\mathfrak{B}$ be a nonempty bounded closed and convex subset of a Banach space $\mathfrak{E}$. Also, let $\mathfrak{W} : \mathfrak{B} \to \mathfrak{B}$ be a continuous map such that*

$$\gamma\left[\vartheta(\mathfrak{W}\mathcal{A}) + \alpha(\vartheta(\mathfrak{W}\mathcal{A}))\right] \leq \gamma[\vartheta(\mathcal{A}) + \alpha(\vartheta(\mathcal{A}))] - \xi\{\gamma[\vartheta(\mathcal{A}) + \alpha(\vartheta(\mathcal{A}))]\}$$

for all $\mathcal{A} \subseteq \mathfrak{B}$, $\gamma \in \Gamma$, $\xi \in \Xi$, where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous map and ϑ is an arbitrary MNC. Then $\mathfrak{W}$ admits at least one fixed point in $\mathfrak{B}$.

Proof The result follows by taking $\Delta(p, q) = p + q$ in Theorem 3.1. □

Theorem 3.4 *Let $\mathfrak{B}$ be a nonempty bounded closed and convex subset of a Banach space $\mathfrak{E}$. Also, let $\mathfrak{W} : \mathfrak{B} \to \mathfrak{B}$ be a continuous map such that*

$$\gamma\left[\vartheta(\mathfrak{W}\mathcal{A})\right] \leq \gamma[\vartheta(\mathcal{A})] - \xi\{\gamma[\vartheta(\mathcal{A})]\}$$

for all $\mathcal{A} \subseteq \mathfrak{B}$, $\gamma \in \Gamma$, $\xi \in \Xi$ where ϑ is an arbitrary MNC. Then $\mathfrak{W}$ possesses at least one fixed point in $\mathfrak{B}$.

Proof By taking $\alpha \equiv 0$ in Theorem 3.3, we obtain the required result. □

Theorem 3.5 *Let $\mathfrak{B}$ be a nonempty bounded closed and convex subset of a Banach space $\mathfrak{E}$. Also, let $\mathfrak{W} : \mathfrak{B} \to \mathfrak{B}$ be a continuous map such that*

$$\vartheta(\mathfrak{W}\mathcal{A}) \leq \lambda\vartheta(\mathcal{A})$$

for all $\mathcal{A} \subseteq \mathfrak{B}$, $\lambda \in [0, 1)$, where ϑ is an arbitrary MNC. Then $\mathfrak{W}$ admits at least one fixed point in $\mathfrak{B}$.

Proof By taking $\gamma(t) = e^t$, $\xi(t) = t - t^\lambda$ for all $t \geq 0$, $\lambda \in [0, 1)$ in Theorem 3.4, we obtain the result. □

Remark 3.6 It can be observed that Theorem 3.5 is DFPT, so it reflects that Theorem 3.1 is a generalization of Theorem 3.5.

Definition 3.7 [10] The ordered pair $(p, q) \in \mathfrak{U} \times \mathfrak{U}$ is said to be a coupled fixed point of a mapping $\mathfrak{G} : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ if $\mathfrak{G}(p, q) = p$ and $\mathfrak{G}(q, p) = q$.

Theorem 3.8 [6] *Suppose that $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ be an MNC in $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_n$, respectively. Moreover, let $\mathfrak{G} : \mathbb{R}_+^n \to \mathbb{R}_+$ be a convex function such that $\mathfrak{G}(p_1, p_2, \ldots, p_n) = 0$ if and only if $p_l = 0$ for $l = 1, 2, \ldots, n$. Then, $\vartheta(\mathfrak{U}) = \mathfrak{G}(\vartheta_1(\mathfrak{U}_1), \vartheta_2(\mathfrak{U}_2), \ldots, \vartheta_n(\mathfrak{U}_n))$ defines an MNC in*

$\mathfrak{K}_1 \times \mathfrak{K}_2 \times \ldots \times \mathfrak{K}_n$, *where* $\mathfrak{U}_l$ *is the natural projection of* $\mathfrak{U}$ *into* $\mathfrak{K}_l$ *for* $l = 1, 2, \ldots, n$.

Example 3.9 [6] Let ϑ be an MNC on $\mathfrak{K}$. Define $\mathfrak{G}(p, q) = p + q,\ p, q \in \mathbb{R}_+$. Then $\mathfrak{G}$ satisfies the properties in Theorem 3.8. Hence, $\vartheta^{cf}(\mathfrak{U}) = \vartheta(\mathfrak{U}_1) + \vartheta(\mathfrak{U}_2)$ is a MNC in $\mathfrak{K} \times \mathfrak{K}$, where $\mathfrak{U}_l,\ l = 1, 2$ denotes the natural projection of $\mathfrak{U}$.

Theorem 3.10 *Let* $\mathfrak{B}$ *be a nonempty bounded closed and convex subset of a Banach space* $\mathfrak{E}$. *Also, assume that* $\mathfrak{W} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ *be a continuous map such that*

$$\begin{aligned}&\gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right)\right)\right)\right]\\&\leq \frac{1}{2}\left\{\gamma\left[\Delta\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right), \alpha\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right)\right)\right)\right]\right.\\&\left.-\xi\left\{\gamma\left[\Delta\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right), \alpha\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right)\right)\right)\right]\right\}\right\}\end{aligned}$$

for all $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathfrak{B}$, *where* ϑ *is an arbitrary MNC and* $\alpha,\ \xi,\ \gamma$ *and* Δ *are as in Theorem* 3.1. *Also, let* $\gamma(p_1 + q_1) \leq \gamma(p_1) + \gamma(q_1),\ p_1, q_1 \geq 0$ *and* $\alpha(p_1 + q_1) \leq \alpha(p_1) + \alpha(q_1),\ p_1, q_1 \geq 0$. *Then* $\mathfrak{W}$ *admits at least a coupled fixed point in* $\mathfrak{B}$.

Proof Consider the mapping $\mathfrak{W}^{cf} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$ by $\mathfrak{W}^{cf}(p, q) = (\mathfrak{W}(p, q), \mathfrak{W}(q, p))$.It is trivial that $\mathfrak{W}^{cf}$ is continuous. Let $\mathcal{A} \subseteq \mathfrak{B} \times \mathfrak{B}$ be nonempty and we have $\vartheta^{cf}(\mathcal{A}) = \vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right)$ is an NMC, where $\mathcal{A}_1, \mathcal{A}_2$ are the natural projections of $\mathcal{A}$ into $\mathfrak{E}$. We obtain

$$\begin{aligned}&\gamma\left[\Delta\left(\vartheta^{cf}(\mathfrak{W}(\mathcal{A})), \alpha\left(\vartheta^{cf}(\mathfrak{W}(\mathcal{A}))\right)\right)\right]\\&\leq \gamma\left[\Delta\left(\vartheta^{cf}\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right) \times \mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right), \alpha\left(\vartheta^{cf}\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\ \times \mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right)\right)\right)\right]\\&= \gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right) + \vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right) + \vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right)\right)\right)\right]\\&\leq \gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right) + \vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right)\right) + \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right)\right)\right)\right]\\&\leq \gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right)\right)\right) + \Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right)\right)\right)\right]\\&\leq \gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right)\right)\right)\right] + \gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_2 \times \mathcal{A}_1\right)\right)\right)\right)\right]\\&\leq \left\{\gamma\left[\Delta\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right), \alpha\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right)\right)\right)\right] - \xi\left\{\gamma\left[\Delta\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right), \alpha\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right)\right)\right)\right]\right\}\right\}\\&= \left\{\gamma\left[\Delta\left(\vartheta^{cf}(\mathcal{A}), \alpha\left(\vartheta^{cf}(\mathcal{A})\right)\right)\right] - \xi\left\{\gamma\left[\Delta\left(\vartheta^{cf}(\mathcal{A}), \alpha\left(\vartheta^{cf}(\mathcal{A})\right)\right)\right]\right\}\right\}.\end{aligned}$$

By Theorem 3.1, we have $\mathfrak{W}^{cf}$ has at least one fixed point in $\mathfrak{B} \times \mathfrak{B}$, i.e., $\mathfrak{W}$ possesses at least one coupled fixed point. ☐

Corollary 3.11 *Let* $\mathfrak{B}$ *be a nonempty bounded closed and convex subset of a Banach space* $\mathfrak{E}$. *Also, let* $\mathfrak{W} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ *is a continuous mapping such that*

$$\begin{aligned}&\gamma\left[\Delta\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right), \alpha\left(\vartheta\left(\mathfrak{W}\left(\mathcal{A}_1 \times \mathcal{A}_2\right)\right)\right)\right)\right]\\&\leq \frac{1}{2}\left\{\gamma\left[\Delta\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right), \alpha\left(\vartheta\left(\mathcal{A}_1\right) + \vartheta\left(\mathcal{A}_2\right)\right)\right)\right]\right\}^{\lambda},\ \lambda \in [0, 1)\end{aligned}$$

for all $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathfrak{B}$, *where* ϑ *is an arbitrary MNC and* $\alpha,\ \gamma$ *and* Δ *are as in Theorem* 3.1. *Also, let* $\gamma(p_1 + q_1) \leq \gamma(p_1) + \gamma(q_1),\ p_1, q_1 \geq 0$ *and* $\alpha(p_1 + q_1) \leq \alpha(p_1) + \alpha(q_1),\ p_1, q_1 \geq 0$. *Then* $\mathfrak{W}$ *has at least a coupled fixed point in* $\mathfrak{B}$.

Proof The result can be obtained by taking $\xi(t) = t - t^{\lambda},\ \lambda \in [0, 1)$ in Theorem 3.10. ☐

Application

The fractional integral of a function $f \in L_1(a, b)$ by another function g of order α is [31],

$$I^{\alpha}_{a+,g} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{(g(x) - g(t))^{1-\alpha}} \mathrm{d}t,\ \alpha > 0,\ -\infty \leq a < b \leq \infty \tag{4.1}$$

which is defined for every continuous function $f(t)$ and for any monotone function $g(t)$ having a continuous derivative.

Analogous to the above operator 4.1, the fractional integral for a continuous function $h(t, s)$ of two variables on $[a, b] \times [a, b]$, by monotone functions g h of order β is defined by

$$I^{\beta}_{a+,g,h} h(X, Y) = \frac{1}{\Gamma(\beta)^2} \int_a^X \int_a^Y \frac{g'(t) h'(s) h(t, s)}{(g(X) - g(t))^{1-\beta} (h(Y) - h(s))^{1-\beta}} \mathrm{d}s \mathrm{d}t \tag{4.2}$$

which is finite, where $\beta > 0,\ \Gamma(Z) = \int_0^{\infty} t^{Z-1} e^{-t} \mathrm{d}t,\ Z > 0$ and $X, Y \in [a, b],\ -\infty \leq a < b \leq \infty$.

Now, we shall check whether the operator (4.2) is strongly continuous semigroup on $C([a,b]\times[a,b],\mathbb{R})$ or not.

It is observed that this operator (4.2) is continuous.

For $h_1(X,Y), h_2(X,Y) \in C([a,b]\times[a,b],\mathbb{R})$ and $k_1, k_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&I^{\beta}_{a+,g,h}\left[k_1h_1(X,Y)+k_2h_2(X,Y)\right] \\
&= \frac{1}{\Gamma(\beta)^2}\int_a^X\int_a^Y \frac{g'(t)h'(s)\left[k_1h_1(X,Y)+k_2h_2(X,Y)\right]}{(g(X)-g(t))^{1-\beta}(h(Y)-h(s))^{1-\beta}}\mathrm{d}s\mathrm{d}t \\
&= \frac{k_1}{\Gamma(\beta)^2}\int_a^X\int_a^Y \frac{g'(t)h'(s)h_1(X,Y)}{(g(X)-g(t))^{1-\beta}(h(Y)-h(s))^{1-\beta}}\mathrm{d}s\mathrm{d}t \\
&\quad + \frac{k_2}{\Gamma(\beta)^2}\int_a^X\int_a^Y \frac{g'(t)h'(s)h_2(X,Y)}{(g(X)-g(t))^{1-\beta}(h(Y)-h(s))^{1-\beta}}\mathrm{d}s \\
&= k_1I^{\beta}_{a+,g,h}h_1(X,Y)+k_2I^{\beta}_{a+,g,h}h_2(x,y)
\end{aligned}$$

so it can be said that the operator (4.2) is linear operator.

Again, for $h_1(X,Y), h_2(X,Y) \geq 0$ we observe that

$$I^{\beta}_{a+,g,h}\left[h_1(X,Y)+h_2(X,Y)\right] \neq I^{\beta}_{a+,g,h}\left[h_1(X,Y)\right]I^{\beta}_{a+,g,h}\left[h_2(X,Y)\right]$$

and

$$I^{\beta}_{a+,g,h}[0] = 0 \neq I.$$

Hence we conclude that the operator (4.2) is not strongly continuous semigroup on $C([a,b]\times[a,b],\mathbb{R})$.

Consider the space $\mathfrak{E} = C(I\times I)$ of all real continuous maps on $I\times I$, where $I=[0,1]$ which is equipped with the norm

$$\| X \| = \sup\{|X(l,m)| : l,m\in I\},\ X\in\mathfrak{E}.$$

Let $\mathfrak{U}$ be a fixed nonempty and bounded subset of $\mathfrak{E}$. For all $X\in\mathfrak{E}$ and $\epsilon>0$, let $\omega(X,\epsilon)$ indicates the modulus of continuity of X, i.e.,

$$\begin{aligned}\omega(X,\epsilon) = \sup\{&|X(T,S)-X(U,V)| : T,S,U,V\in I,\\ &|T-U|\leq\epsilon, |S-V|\leq\epsilon\}.\end{aligned}$$

Further, let

$$\omega(\mathfrak{U},\epsilon) = \sup\{\omega(X,\epsilon) : X\in\mathfrak{U}\},$$

and

$$\omega_0(\mathfrak{U}) = \lim_{\epsilon\to 0}\omega(\mathfrak{U},\epsilon).$$

Similar to [7], it can be proved that the function ω_0 is an MNC in the space $\mathfrak{E}$.

In this part, the solvability of the following generalized fractional order integral equation is studied:

$$\begin{aligned}&\pi(t,s)\\ &= H\left(t,s,\pi(t,s), \int_0^t\int_0^s \frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right),\end{aligned} \tag{4.3}$$

where $0<\alpha<1,\ t,s\in I=[0,T],\ T>0.$

Assumptions:

(1) $H : I\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous mapping satisfying

$$|H(t,s,p,q)-H(t,s,l,m)| \leq A|p-l| + B|q-m|$$

for some nonnegative constants $A,\ B$ with $A\in[0,1)$, where $t,s\in I; p,q,l,m\in\mathbb{R}$.

(2) The functions $g,h : I\to\mathbb{R}_+$ are C^1 and nondecreasing. Also, $g',h'\geq 0$.

(3) $k : I\times I\times I\times I\times\mathbb{R}\to\mathbb{R}$ is continuous.

(4) Let

$$K = \sup\{|k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))| : t,s,\sigma,\varsigma\in I; x\in C(I\times I)\}$$

and

$$\hat{H} = \sup\{|H(t,s,0,0)| : t,s\in I\}$$

and let there exists $r_0>0$ satisfying $Ar_0 + \frac{BK}{\alpha^2}(h(T)-h(0))^{\alpha}(g(T)-g(0))^{\alpha} + \hat{H} \leq r_0$. Let $B_{r_0} = \left\{x\in\mathfrak{E} : \| x \| \leq r_0\right\}$.

Theorem 4.1 *Under the hypothesis (1)-(4), equation (4.3) has at least one solution in* $\mathfrak{E}$.

Proof For $\pi\in\mathfrak{E}$, let the operator $\mathfrak{Q}$ be defined on $\mathfrak{E}$ as follows:

$$\begin{aligned}&(\mathfrak{Q}\pi)(t,s)\\ &= H\left(t,s,\pi(t,s), \int_0^t\int_0^s \frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right),\end{aligned}$$

where $t,s\in I$.

Let $t,s\in I$ be fixed and $\{t_n\}$ and $\{s_n\}$ be sequences in I such that $t_n\to t$ and $s_n\to s$ as $n\to\infty$. Without loss of generality, we can choose $t_n\geq t$ and $s_n\geq s$. Then

$$\begin{aligned}&\left|(\mathfrak{Q}\pi)(t_n,s_n)-(\mathfrak{Q}\pi)(t,s)\right|\\
&\leq A\left|\pi(t_n,s_n)-\pi(t,s)\right|\\
&\quad + B\left|\int_0^{t_n}\int_0^{s_n}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_n)-g(\varsigma)\right)^{1-\alpha}\left(h(t_n)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right.\\
&\quad\left. - \int_0^t\int_0^s\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right|.\end{aligned}$$

Now,

$$\begin{aligned}
&\Big|\int_0^{t_n}\int_0^{s_n}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\Big|\\
&\leq\Big|\int_0^{t_n}\int_0^{s_n}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\Big|\\
&\quad+\Big|\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\Big|\\
&\quad+\Big|\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\Big|\\
&=\mathfrak{I}_n+\mathfrak{I}\mathfrak{I}_n+\mathfrak{I}\mathfrak{I}\mathfrak{I}_n,
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{I}_n&=\Big|\int_0^{t_n}\int_0^{s_n}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\\
&-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\Big|\\
&=\Big|\int_t^{t_n}\int_0^{s_n}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\\
&+\int_0^{t}\int_s^{s_n}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\Big(g(s_n)-g(\varsigma)\Big)^{1-\alpha}\Big(h(t_n)-h(\sigma)\Big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\Big|\\
&\leq\frac{K}{\alpha^2}\Big(h(t_n)-h(t)\Big)^{\alpha}\Big(g(s_n)-g(0)\Big)^{\alpha}-\frac{K}{\alpha^2}\Big[\Big(h(t_n)-h(t)\Big)^{\alpha}-\Big(h(t_n)-h(0)\Big)^{\alpha}\Big]\Big(g(s_n)-g(s)\Big)^{\alpha}.
\end{aligned}$$

As $n\to\infty$, continuity of g and h yields that $\mathfrak{I}_n\to 0$. Again,

$$\begin{aligned}
\mathfrak{I}\mathfrak{I}_n&=\left|\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\big(g(s_n)-g(\varsigma)\big)^{1-\alpha}\big(h(t_n)-h(\sigma)\big)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t_n,s_n,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right|\\
&\leq-K\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)\mathrm{d}\varsigma\mathrm{d}\sigma}{\big(g(s_n)-g(\varsigma)\big)^{1-\alpha}\big(h(t_n)-h(\sigma)\big)^{1-\alpha}}+K\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)\mathrm{d}\varsigma\mathrm{d}\sigma}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\\
&=\frac{K}{\alpha^2}\big[\big(g(s_n)-g(s)\big)^{\alpha}\big(h(t_n)-h(0)\big)^{\alpha}+\big(g(s_n)-g(0)\big)^{\alpha}\big(h(t_n)-h(t)\big)^{\alpha}-\big(g(s_n)-g(s)\big)^{\alpha}\big(h(t_n)-h(t)\big)^{\alpha}\big].
\end{aligned}$$

Taking $n\to\infty$ and using the continuity of g and h, it is observed that $\mathfrak{I}\mathfrak{I}_n\to 0$. Finally,

$$\mathfrak{I}\mathfrak{I}\mathfrak{I}_n = \left|\int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t_n, s_n, \sigma, \varsigma, \pi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} d\varsigma d\sigma - \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \pi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma\right|$$
$$\leq \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)\left|k(t_n, s_n, \sigma, \varsigma, \pi(\sigma, \varsigma)) - k(t, s, \sigma, \varsigma, \pi(\sigma, \varsigma))\right|}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma.$$

Since k is a continuous function, therefore it is observed that $\mathfrak{I}\mathfrak{I}\mathfrak{I}_n \to 0$, as $n \to \infty$.

Thus, $\pi(t, s) \in \mathfrak{E}$ gives $\mathfrak{L}\pi \in \mathfrak{E}$. So, the mapping $\mathfrak{L} : \mathfrak{E} \to \mathfrak{E}$ is well defined.

Let $B_{r_0} = \left\{\pi \in \mathfrak{E} : \| \pi \| \leq r_0\right\}$. Also, let $\pi, \varpi \in B_{r_0}$. Then, for all $t, s \in I$ it is observed that

$$|(\mathfrak{L}\pi)(t, s)|$$
$$\leq \left|H\left(t, s, \pi(t, s), \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \pi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} d\varsigma d\sigma\right) - H(t, s, 0, 0)\right| + \hat{H}$$
$$\leq A|\pi(t, s)| + BK \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma + \hat{H}$$
$$\leq A r_0 + \frac{BK}{\alpha^2}(g(T) - g(0))^{\alpha}(h(T) - h(0))^{\alpha} + \hat{H}.$$
$$\leq r_0.$$

Therefore, $\mathfrak{L}(B_{r_0}) \subseteq B_{r_0}$, i.e., $\mathfrak{L} : B_{r_0} \to B_{r_0}$ is well defined.

Let $\pi, \varpi \in B_{r_0}$ be such that $\| \pi - \varpi \| \leq \epsilon$ where $\epsilon > 0$. For all $t, s \in I$,

$$|(\mathfrak{L}\pi)(t, s) - (\mathfrak{L}\varpi)(t, s)|$$
$$= \left|H\left(t, s, \pi(t, s), \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \pi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma\right)\right.$$
$$\left. -H\left(t, s, \varpi(t, s), \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \varpi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma\right)\right|$$
$$\leq A|\pi(t, s) - \varpi(t, s)|$$
$$+ B \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)|k(t, s, \sigma, \varsigma, \pi(\sigma, \varsigma)) - k(t, s, \sigma, \varsigma, \varpi(\sigma, \varsigma))|}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} d\varsigma d\sigma$$
$$\leq A \| \pi - \varpi \| + \frac{Bk_\epsilon}{\alpha^2}(g(T) - g(0))^{\alpha}(h(T) - h(0))^{\alpha},$$

where

$$k_\epsilon = \sup \left\{ \begin{array}{l} |k(t, s, , \sigma, \varsigma, \pi) - k(t, s, \sigma, \varsigma, \varpi)| : t, s, \sigma, \varsigma \in I, \\ |\pi - \varpi| \leq \epsilon, |\pi| \leq r_0, |\varpi| \leq r_0. \end{array} \right\}$$

Since k is an uniformly continuous function on $I \times I \times I \times I \times [-r_0, r_0]$, therefore $k_\epsilon \to 0$, as $\epsilon \to 0$. Therefore, $\| \mathfrak{L}\pi - \mathfrak{L}\varpi \| \to 0$, as $\epsilon \to 0$, i.e., $\mathfrak{L}$ is continuous on B_{r_0}.Let $P \subseteq B_{r_0}$ be nonempty. For an arbitrary $\epsilon > 0$, take $\pi(t, s) \in P$ and $t, s, t_1, s_1 \in I$ such that $\left|t - t_1\right| \leq \epsilon$, $\left|s - s_1\right| \leq \epsilon$. Without loss of generality, it can be taken $t_1 \geq t$, $s_1 \geq s$. Now,

$$
\begin{aligned}
&\left|(\mathfrak{L}\pi)(t_1,s_1)-(\mathfrak{L}\pi)(t,s)\right| \\
&=\left|H\left(t_1,s_1,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right.\\
&\left.-H\left(t,s,\pi(t,s),\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right|\\
&\leq\left|H\left(t_1,s_1,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}d\varsigma d\sigma\right)\right.\\
&\left.-H\left(t,s,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right|\\
&+\left|H\left(t,s,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right.\\
&\left.-H\left(t,s,\pi(t,s),\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right|.\\
&=\mathfrak{I}+\mathfrak{I}\mathfrak{I}
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\left|\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right|\\
&\leq\frac{K}{\alpha^2}(g(T)-g(0))^{\alpha}(h(T)-h(0))^{\alpha}=Q(say).
\end{aligned}
$$

Let

$$
C(H,\epsilon)=\sup\left\{\begin{array}{l}\left|H(t,s,,\pi,l)-H\left(t_1,s_1,\pi,l\right)\right|\,:\,t,s,t_1,s_1\in I,\\ \left|t-t_1\right|\leq\epsilon,\left|s-s_1\right|\leq\epsilon,\ |\pi|\leq r_0,|l|\leq Q.\end{array}\right\}.
$$

Therefore,

$$
\begin{aligned}
\mathfrak{I}=&\left|H\left(t_1,s_1,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right.\\
&\left.-H\left(t,s,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\right|\\
&\leq C(H,\epsilon).
\end{aligned}
$$

By the uniform continuity of H in $I\times I\times[-r_0,r_0]\times[-Q,Q]$, we have $\lim_{\epsilon\to 0}C(H,\epsilon)=0$.Again,

$$
\begin{aligned}
\mathfrak{I}\mathfrak{I}=&|\,H\left(t,s,\pi(t_1,s_1),\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)\\
&-H\left(t,s,\pi(t,s),\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right)|\\
&\leq A\left|\pi(t_1,s_1)-\pi(t,s)\right|\\
&+B\left|\int_0^{t_1}\int_0^{s_1}\frac{g'(\varsigma)h'(\sigma)k(t_1,s_1,\sigma,\varsigma,\pi(\sigma,\varsigma))}{\left(g(s_1)-g(\varsigma)\right)^{1-\alpha}\left(h(t_1)-h(\sigma)\right)^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma-\int_0^{t}\int_0^{s}\frac{g'(\varsigma)h'(\sigma)k(t,s,\sigma,\varsigma,\pi(\sigma,\varsigma))}{(g(s)-g(\varsigma))^{1-\alpha}(h(t)-h(\sigma))^{1-\alpha}}\mathrm{d}\varsigma\mathrm{d}\sigma\right|.
\end{aligned}
$$

Let

$$C(k,\epsilon) = \sup \left\{ \begin{array}{l} \left| k\left(t_1, s_1, \sigma, \varsigma, \pi\right) - k\left(t_2, s_2, \sigma, \varsigma, \pi\right) \right| \,:\, t, s, t_1, s_1 \in I, \\ \left| t_1 - t \right| \le \epsilon,\ \left| s_1 - s \right| \le \epsilon,\ \pi \in [-r_0, r_0] \end{array} \right\},$$

$$C(h,\epsilon) = \sup \left\{ \left| h(t_1) - h(t) \right| \,:\, t, t_1 \in I,\ \left| t_1 - t \right| \le \epsilon \right\}$$

and

$$C(g,\epsilon) = \sup \left\{ \left| g(t_1) - g(t) \right| \,:\, t, t_1 \in I,\ \left| t_1 - t \right| \le \epsilon \right\}.$$

On the other hand,

$$\begin{aligned}
&\left| \int_0^{t_1} \int_0^{s_1} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma - \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \pi(\sigma,\varsigma))}{(g(s) - g(\varsigma))^{1-\alpha} (h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma \right| \\
&\quad \le \left| \int_0^{t_1} \int_0^{s_1} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma - \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma \right| \\
&\quad + \left| \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} d\varsigma d\sigma - \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{(g(s) - g(\varsigma))^{1-\alpha} (h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma \right| \\
&\quad + \left| \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{(g(s) - g(\varsigma))^{1-\alpha} (h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma - \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \pi(\sigma,\varsigma))}{(g(s) - g(\varsigma))^{1-\alpha} (h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma \right| \\
&= \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_1 &= \left| \int_0^{t_1} \int_0^{s_1} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma - \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma \right| \\
&\le \frac{K}{\alpha^2} \left\{ \left(g(s_1) - g(0)\right)^{\alpha} \left(h(t_1) - h(t)\right)^{\alpha} - \left(g(s_1) - g(s)\right)^{\alpha} \left(h(t_1) - h(t)\right)^{\alpha} + \left(g(s_1) - g(s)\right)^{\alpha} \left(h(t_1) - h(0)\right)^{\alpha} \right\} \\
&\le \frac{K}{\alpha^2} \left\{ \left(g(s_1) - g(0)\right)^{\alpha} \left(h(t_1) - h(t)\right)^{\alpha} + \left(g(s_1) - g(s)\right)^{\alpha} \left(h(t_1) - h(0)\right)^{\alpha} \right\} \\
&\le \frac{2K\{C(g,\epsilon)C(h,\epsilon)\}^{\alpha}}{\alpha^2}, \\
\mathcal{V}_2 &= \left| \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{\left(g(s_1) - g(\varsigma)\right)^{1-\alpha} \left(h(t_1) - h(\sigma)\right)^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma - \int_0^{t} \int_0^{s} \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma,\varsigma))}{(g(s) - g(\varsigma))^{1-\alpha} (h(t) - h(\sigma))^{1-\alpha}} \mathrm{d}\varsigma \mathrm{d}\sigma \right| \\
&\le \frac{K}{\alpha^2} \left\{ \left(g(s_1) - g(s)\right)^{\alpha} \left(h(t_1) - h(0)\right)^{\alpha} + \left(g(s_1) - g(0)\right)^{\alpha} \left(h(t_1) - h(t)\right)^{\alpha} - \left(g(s_1) - g(s)\right)^{\alpha} \left(h(t_1) - h(t)\right)^{\alpha} \right\} \\
&\le \frac{K}{\alpha^2} \left\{ \left(g(s_1) - g(s)\right)^{\alpha} \left(h(t_1) - h(0)\right)^{\alpha} + \left(g(s_1) - g(0)\right)^{\alpha} \left(h(t_1) - h(t)\right)^{\alpha} \right\} \\
&\le \frac{2K\{C(g,\epsilon)C(h,\epsilon)\}^{\alpha}}{\alpha^2}
\end{aligned}$$

$$\mathcal{V}_3 = \left| \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t_1, s_1, \sigma, \varsigma, \pi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} d\varsigma d\sigma - \int_0^t \int_0^s \frac{g'(\varsigma)h'(\sigma)k(t, s, \sigma, \varsigma, \pi(\sigma, \varsigma))}{(g(s) - g(\varsigma))^{1-\alpha}(h(t) - h(\sigma))^{1-\alpha}} d\varsigma d\sigma \right|$$
$$\leq \frac{C(k, \epsilon)}{\alpha^2}(g(s) - g(0))^\alpha (h(t) - h(t))^\alpha$$
$$\leq \frac{C(k, \epsilon)\{C(g, \epsilon)C(h, \epsilon)\}^\alpha}{\alpha^2}.$$

Then,

$$\mathfrak{I}\mathfrak{I} \leq A\omega(P, \epsilon) + B\left\{ \frac{4K\{C(g, \epsilon)C(h, \epsilon)\}^\alpha}{\alpha^2} + \frac{C(k, \epsilon)\{C(g, \epsilon)C(h, \epsilon)\}^\alpha}{\alpha^2} \right\}.$$

Therefore,

$$\omega(\mathfrak{L}P, \epsilon) \leq C(H, \epsilon) + A\omega(P, \epsilon) + B \left\{ \frac{4K\{C(g, \epsilon)C(h, \epsilon)\}^\alpha}{\alpha^2} + \frac{C(k, \epsilon)\{C(g, \epsilon)C(h, \epsilon)\}^\alpha}{\alpha^2} \right\}.$$

Since H, g, h are all continuous, therefore for all $\epsilon \to 0$ assumption (1) and Theorem 3.5 gives $\mathfrak{L}$ admits at least one fixed point in $P \subseteq B_{d_0} \subseteq \mathfrak{E}$. □

Example 4.2 We consider

$$\pi(t, s) = \frac{ts(1 + \pi(t, s))}{1 + ts} + \int_0^t \int_0^s \frac{\pi^2(\sigma, \varsigma)}{(t - \sigma)^{\frac{1}{2}}(s - \varsigma)^{\frac{1}{2}}\left(1 + \pi^2(\sigma, \varsigma)\right)} d\varsigma d\sigma \tag{4.4}$$

for all $t, s, \sigma, \varsigma \in [0, 1] = I$.

Here,

$$g(t) = h(t) = t, \ \ \alpha = \frac{1}{2}; k(t, s, \sigma, \varsigma, \pi) = \frac{\pi^2}{1 + \pi^2}$$

and

$$H(t, s, \pi, q) = \frac{ts(1 + \pi)}{1 + ts} + q.$$

For all $t, s \in I$ and $p, q, l, m \in \mathbb{R}$,

$$|H(t, s, p, q) - H(t, s, l, m)| \leq \frac{ts}{1 + ts}|p - l| + |q - m|.$$

Here, $A = \frac{1}{2}$ and $B = 1$. The functions $g, h : I \to \mathbb{R}_+$ are C^1 nondecreasing. Also, $g', h' > 0$. The functions k and H are continuous and $K = 1$ and $\hat{H} = \frac{1}{2}$.

Also, for $r_0 = 9$ the inequality $\frac{r_0}{2} + 4(1-0)^{\frac{1}{2}}(1-0)^{\frac{1}{2}} + \frac{1}{2} \leq r_0$ is satisfied. Thus, for $r_0 = 9$ assumptions $(1) - -(4)$ of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1 we conclude that Eq. () possesses at least one solution in $C([0, 1] \times [0, 1])$.

Conclusion

In this work, we generalized DFPT by MNC and a new contraction operator and also we have established the corresponding coupled fixed point theorem. With the help of this generalized DFPT, we have established the existence of solution of a integral equation with generalized fractional integral of two variables and finally illustrate the results with the help of an example.

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