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An existence result for an infinite system of implicit fractional integral equations via generalized Darbo's fixed point theorem

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Abstract
In the current article we obtain the extension of Darbo's fixed point theorem (DFPT), and apply this theorem to prove the existence of solution of an infinite system of implicit fractional integral equations. We, besides that, justify the results with the help of an example. The advantage of the proposed fixed point theory is that the requirement of the compactness of the domain is relaxed which is essential in some fixed point theorems. Also, we have applied it to integral equation involving fractional integral by another function which is a generalization of many fixed point theorems as well as fractional integral equations.

Keywords Implicit fractional integral equation (IFIE) · Measure of noncompactness (MNC) · Darbo's fixed point theorem

Mathematics Subject Classification 45G05 · 26A33 · 74H20

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1 Introduction

The integral equations have multiple practical applications in defining specific real-world problems and different types of real life situations, i.e., in laws of physics, the theory of radioactive transmission, the theory of statistical mechanics, and the cytotoxic activity the integral equations are applied for instance see (Boffi and Spiga 1983; Case and Zweifel 1967; Chandrasekhar 1960; Hu et al. 1989; Kelly 1982). Kuratowski (1930) was initiated the notion measure of noncompactness in metric spaces (one can refer Banaś and Goebel 1980; Banaś and Mursaleen 2014 for the detailed on MNC). Darbo (1955) was the first person to implemented the measure of noncompactness to generalized the Banach fixed point theorem for Banach spaces. It is familiar as Darbo fixed point theorem. Recently, numbers of articles published in connection with the solvability of different types of integral equations, nonlinear integral equations, functional integral equations, differential equations, infinite systems of integral equations using different fixed point theorems and measure of noncompactness (MNC) in Banach spaces (readers can consult the papers Agarwal and O'Regan 2004; Aghajani et al. 2014; Alotaibi et al. 2015; Banaei et al. 2020; Çakan and Özdemir 2017; Das et al. 2019; Deep et al. 2020; Hazarika et al. 2019, 2018, 2019, ?, 2021; Işik et al. 2020; Kazemi and Ezzati 2016; Mursaleen and Mohiuddine 2012; Mohammadi et al. 2020; Nashine et al. 2018, 2017; Rabbani et al. 2019; Srivastava et al. 2018 and references therein).

Fractional calculus is the study of the derivatives as well as integrals of arbitrary order using Gamma function. In the 16th century the concepts of fractional calculus was adopted. Over the 19th and early 20th centuries, the theory and applications of fractional calculus developed greatly, and countless contributors have provided interpretations for fractional derivatives and integrals. In many branches of mathematics like porous media, viscoelasticity and electrochemistry etc the Erdélyi–Kober fractional integrals are used (one can consulted Chandrasekhar 1960; Erdélyi 1950; Hilfer 2000; Kober 1940; Pagnini 2012 among others). Due to the importance of integral equations of fractional order it has become essential to study such type of equations. Many authors considered differential and integral equations involving Erdélyi–Kober fractional operator, few number of articles we mentioned here (Arab et al. 2020; Darwish and Sadarangani 2015; Darwish 2016, 2011; Mollapourasl and Ostadi 2015; Rabbani et al. 2020; Samko et al. 1993). Recently, some authors have studied time-fractional diffusion problems, fractional reaction-subdiffusion problem etc (see, for instance, Nikan et al. 2021; Nikan and Avazzadeh 2021; Nikan et al. 2021a, b).

Mohammadi et al. (2020) have established a new generalization of Darbo's fixed point theorem with the help of a newly defined contraction operator and applied it to a system of integral equations. Motivated by their work, we have introduced a more general condensing operator to established a generalization of the fixed point theorem which was investigated in Mohammadi et al. (2020) and finally apply it to a fractional integral equation by another function.

The main contribution of this article is that we have extended the results of the article Mohammadi et al. (2020) to establish a generalization of Darbo's fixed point and apply it to obtain the existence of the solution of the following infinite system of integral equations

$$z_n(x) = \mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x, w, z(w))}{(g(x) - g(w))^{1-\alpha}}\mathrm{d}w\right), \ n \in \mathbb{N}.$$

The above system involves a fractional integral by another function which is the generalization of many other fractional integral equations.

So it can be seen that our work a generalization of many other application of Darbo's fixed point theorem on a system of fractional integral equations.

Consider $(E, \| \,.\, \|)$ is a Banach space. Let $B[\theta, r]$ be a closed ball in E centered at θ and with radius r. If $\mathfrak{X}$ is a nonempty subset of E, then by $\bar{\mathfrak{X}}$ and Conv$\mathfrak{X}$ we denote the closure and convex closure of $\mathfrak{X}$, respectively. Moreover, let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. We denote $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$.

Definition 1 (Banaś and Goebel 1980) A function $\vartheta : \mathfrak{M}_E \to \mathbb{R}_+$ is called a MNC in E if it satisfies the below stated conditions:

(i) for all $\mathfrak{Y} \in \mathfrak{M}_E$, we have $\vartheta(\mathfrak{Y}) = 0$ implies that $\mathfrak{Y}$ is precompact.
(ii) the family $\ker \vartheta = \{\mathfrak{Y} \in \mathfrak{M}_E : \vartheta(\mathfrak{Y}) = 0\}$ is nonempty and $\ker \vartheta \subset \mathfrak{N}_E$.
(iii) $\mathfrak{Y} \subseteq \mathfrak{Z} \implies \vartheta(\mathfrak{Y}) \le \vartheta(\mathfrak{Z})$.
(iv) $\vartheta\left(\bar{\mathfrak{Y}}\right) = \vartheta(\mathfrak{Y})$.
(v) $\vartheta(Conv\mathfrak{Y}) = \vartheta(\mathfrak{Y})$.
(vi) $\vartheta(\lambda\mathfrak{Y} + (1-\lambda)\mathfrak{Z}) \le \lambda\vartheta(\mathfrak{Y}) + (1-\lambda)\vartheta(\mathfrak{Z})$ for $\lambda \in [0, 1]$.
(vii) if $\mathfrak{Y}_n \in \mathfrak{M}_E,\ \mathfrak{Y}_n = \bar{\mathfrak{Y}}_n,\ \mathfrak{Y}_{n+1} \subset \mathfrak{Y}_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \vartheta(\mathfrak{Y}_n) = 0$ then $\bigcap_{n=1}^{\infty} \mathfrak{Y}_n \neq \phi$.

The family $\ker \vartheta$ is said to be the *kernel of measure* ϑ. Observe that the intersection set $\mathfrak{Y}_\infty$ from (vii) is a member of the family $\ker \vartheta$. Also since $\vartheta(\mathfrak{Y}_\infty) \le \vartheta(\mathfrak{Y}_n)$ for any n, we infer that $\vartheta(\mathfrak{Y}_\infty) = 0$. This gives $\mathfrak{Y}_\infty \in \ker\vartheta$.

The Hausdorff MNC for a bounded set $\mathfrak{S}$ is defined as

$$\chi(\mathfrak{S}) = \inf\{\epsilon > 0 : \mathfrak{S} \text{ has finite } \epsilon - \text{net in } \mathfrak{X}\}.$$

The Hausdorff MNC χ in the Banach space $\left(c_0, \| \,.\, \|_{c_0}\right)$ can be formulated as follows (see Banaś and Goebel 1980):

$$\chi_{c_0}\left(\hat{D}\right) = \lim_{n\to\infty}\left[\sup_{u\in\hat{D}}\left(\max_{k\ge n} | u_k |\right)\right], \tag{1}$$

where $u = (u_i)_{i=1}^{\infty} \in c_0$ and $\hat{D} \in \mathcal{M}_{c_0}$.

In the Banach space $\left(\ell_1, \| \,.\, \|_{\ell_1}\right)$, the Hausdorff MNC χ is defined as follows (see Banaś and Goebel 1980):

$$\chi_{\ell_1}\left(\hat{D}\right) = \lim_{n\to\infty}\left[\sup_{u\in\hat{D}}\left(\sum_{k=n}^{\infty} | u_k |\right)\right], \tag{2}$$

where $u = (u_i)_{i=1}^{\infty} \in \ell_1$ and $\hat{D} \in \mathcal{M}_{\ell_1}$.

Let us denote by $C(I, c_0),\ I = [a, \tau],\ a \ge 0,\ \tau > 0$ the space of all continuous functions on I with values in c_0. Then $C(I, c_0)$ is also a Banach space with norm $\| x(t) \|_{C(I,c_0)} = \sup\left\{\| x(t) \|_{c_0} : t \in I\right\}$, where $x(t) \in C(I, c_0)$.

For any non-empty bounded subset $\hat{E}$ of $C(I, c_0)$ and $t \in I$, let $\hat{E}(t) = \left\{x(t) : x \in \hat{E}\right\}$. Now, using (1), we conclude that the Hausdorff MNC for $\hat{E} \subset C(I, c_0)$ can be defined as

$$\chi_{C(I,c_0)}(\hat{E}) = \sup\left\{\chi_{c_0}(\hat{E}(t)) : t \in I\right\}.$$

Similarly, we can define $C(I, \ell_1)$, the space of all continuous functions defined on I with values in ℓ_1. Then $C(I, \ell_1)$ is also a Banach space with the norm $\| x(t) \|_{C(I,\ell_1)} = \sup\left\{\| x(t) \|_{\ell_1} : t \in I\right\}$, where $x(t) \in C(I, \ell_1)$.

Now, using (2), we conclude that the Hausdorff MNC for $\hat{E} \subset C(I, \ell_1)$ can be defined by

$$\chi_{C(I,\ell_1)}(\hat{E}) = \sup\left\{\chi_{\ell_1}(\hat{E}(t)) : t \in I\right\}.$$

Definition 2 (Banaś and Goebel 1980) Let $\mathfrak{X}$ be a nonempty subset of a Banach space E and $\mathfrak{T} : \mathfrak{X} \to E$ is a continuous operator transforming bounded subset of $\mathfrak{X}$ to bounded ones. We say that $\mathfrak{T}$ satisfies the Darbo condition with a constant k with respect to measure ϑ provided $\vartheta(\mathfrak{T}\mathfrak{Y}) \leq k\vartheta(\mathfrak{Y})$ for each $\mathfrak{Y} \in \mathfrak{M}_E$ such that $\mathfrak{Y} \subset \mathfrak{X}$.

Theorem 1 (Agarwal and O'Regan 2004, Schauder) *Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a Banach space $\bar{E}$. Then every compact, continuous mapping $\mathfrak{T} : \mathfrak{D} \to \mathfrak{D}$ has minimum of one fixed point.*

Theorem 2 (Darbo 1955, Darbo) *Let $\mathfrak{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\bar{E}$. Let $\mathfrak{S} : \mathfrak{Z} \to \mathfrak{Z}$ be a continuous mapping. Assume that there is a constant $k \in [0, 1)$ such that*

$$\vartheta(\mathfrak{S}\mathfrak{M}) \leq k\vartheta(\mathfrak{M}), \ \mathfrak{M} \subseteq \mathfrak{Z}.$$

Then $\mathfrak{S}$ has a fixed point.

From the above to two fixed point theorems it is clear that the Darbo fixed point theorem is more effective than Schauder fixed point theorem. In this connection we mentioned the following remark.

Remark 1 We have generalized Darbo's fixed point theorem using a new contraction operator which involves MNC to study operators whose properties can be characterized as being intermediate between those of contraction and compact mapping. The main advantage of this generalization using MNC is that the compactness of domain of the operator which is essential in Schauder's theorem has been relaxed.

We use the following concepts to established the extension of Darbo's fixed point theorem.

Definition 3 (Hazarika et al. 2018) Let $\mathcal{F}$ be the class of all functions $J : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(1) $\max\{\iota, \varpi\} \leq J(\iota, \varpi)$ for $\iota, \varpi \geq 0$.
(2) J is continuous and nondecreasing.
(3) $J(\iota + \varpi, \iota_1 + \varpi_1) \leq J(\iota, \iota_1) + J(\varpi, \varpi_1)$.

For example, $J(\iota, \varpi) = \iota + \varpi$.

Definition 4 (Mohammadi et al. 2020) Suppose Δ is the set of all functions $\mathfrak{W} : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(1) $\mathfrak{W}$ is continuous strictly increasing function.
(2) $\lim_{n\to\infty} \mathfrak{W}(\varsigma_n) = -\infty$ if and only if $\lim_{n\to\infty} \varsigma_n = 0$ for all $\{\varsigma_n\} \subseteq \mathbb{R}_+$.

For example,

(a) $\mathfrak{W}_1(\varsigma) = \ln(\varsigma)$
(b) $\mathfrak{W}_2(\varsigma) = 1 - \frac{1}{\varsigma^p}, \ p > 0$

(c) $\mathfrak{W}_3(\varsigma) = 1 - \frac{1}{e^{\varsigma}-1}$
(d) $\mathfrak{W}_4(\varsigma) = \frac{1}{e^{-\varsigma}-e^{\varsigma}}$

belongs to Δ.

Definition 5 (Mohammadi et al. 2020) Let Θ be the class of all functions $\mathcal{V} : \mathbb{R} \to \mathbb{R}$ satisfying:

(1) $\lim_{n\to\infty} \mathcal{V}^n(\varsigma) = -\infty$ for all $\varsigma > 0$.
(2) $\mathcal{V}(\varsigma) < \varsigma$ for all $\varsigma \geq 0$.
(3) $\mathcal{V}$ is increasing and continuous.

For example,

(i) $\mathcal{V}_1(\varsigma) = \varsigma - a,\ a > 0$ belongs to Θ,
(ii) $\mathcal{V}_2(\varsigma) = \varsigma^3 - 1,\ \varsigma \leq 1$ belongs to Θ,
(iii) $\mathcal{V}_3(\varsigma) = \varsigma^{\frac{1}{3}} - 1,\ \varsigma \geq 1$ belongs to Θ.

2 Generalized fixed point theorems

Theorem 3 *Let $\mathfrak{P}$ be a nonempty, bounded, closed and convex (BCC) subset of a Banach space $\mathfrak{H}$. Also $\mathfrak{L} : \mathfrak{P} \to \mathfrak{P}$ is a continuous function such that*

$$\mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\mathcal{Q}\right), \phi\left(\vartheta\left(\mathfrak{L}\mathcal{Q}\right)\right)\right)\right] \leq \mathcal{V}\left\{\mathfrak{W}\left[J\left(\vartheta\left(\mathcal{Q}\right), \phi\left(\vartheta\left(\mathcal{Q}\right)\right)\right)\right]\right\} \tag{3}$$

for all $\mathcal{Q} \subseteq \mathfrak{P}$, $\mathfrak{W} \in \Delta$, $\mathcal{V} \in \Theta$, $J \in \mathcal{F}$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, where ϑ is an arbitrary MNC. Then $\mathfrak{L}$ has minimum of one fixed point in $\mathfrak{P}$.

Proof Define a sequence $(\mathfrak{P}_q)$ such that $\mathfrak{P}_1 = \mathfrak{P}$ and $\mathfrak{P}_{q+1} = \mathrm{Conv}(\mathfrak{L}\mathfrak{P}_q)$ for $q \geq 1$. Also $\mathfrak{L}\mathfrak{P}_1 = \mathfrak{L}\mathfrak{P} \subseteq \mathfrak{P} = \mathfrak{P}_1$, $\mathfrak{P}_2 = \mathrm{Conv}(\mathfrak{L}\mathfrak{P}_1) \subseteq \mathfrak{P} = \mathfrak{P}_1$; therefore, consequently, through extending such framework, we obtain $\mathfrak{P}_1 \supseteq \mathfrak{P}_2 \supseteq \cdots \supseteq \mathfrak{P}_q \supseteq \mathfrak{P}_{q+1} \supseteq \cdots$.

If there exists $\hat{q} \in \mathbb{N}$ satisfying $\vartheta(\mathfrak{P}_{\hat{q}}) = 0$ then $\mathfrak{P}_{\hat{q}}$ is compact. By Schauder's fixed point theorem we conclude that $\mathfrak{L}$ has a fixed point.

If $\vartheta(\mathfrak{P}_q) > 0$ for all $q \in \mathbb{N}$, clearly $\left\{\vartheta(\mathfrak{P}_q)\right\}$ is nonnegative, decreasing and bounded below sequence.

Also, $\vartheta(\mathfrak{P}_{q+1}) = \vartheta(Conv\left(\mathfrak{L}\mathfrak{P}_q\right)) = \vartheta(\mathfrak{L}\mathfrak{P}_q)$ and by (3), we have

$$\begin{aligned}
&\mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{P}_{q+1}\right), \phi\left(\vartheta\left(\mathfrak{P}_{q+1}\right)\right)\right)\right] \\
&= \mathfrak{W}\left[J\left(\vartheta\left(\mathrm{Conv}\left(\mathfrak{L}\mathfrak{P}_q\right)\right), \phi\left(\vartheta\left(\mathrm{Conv}\left(\mathfrak{L}\mathfrak{P}_q\right)\right)\right)\right)\right] \\
&= \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\mathfrak{P}_q\right), \phi\left(\vartheta\left(\mathfrak{L}\mathfrak{P}_q\right)\right)\right)\right] \\
&\leq \mathcal{V}\left\{\mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{P}_q\right), \phi\left(\vartheta\left(\mathfrak{P}_q\right)\right)\right)\right]\right\} \\
&\leq \mathcal{V}^2\left\{\mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{P}_{q-1}\right), \phi\left(\vartheta\left(\mathfrak{P}_{q-1}\right)\right)\right)\right]\right\} \\
&\vdots \\
&\leq \mathcal{V}^q\left\{\mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{P}_1\right), \phi\left(\vartheta\left(\mathfrak{P}_1\right)\right)\right)\right]\right\}.
\end{aligned}$$

As $q \to \infty$ and applying the Definition 5, we get

$$\lim_{q\to\infty} \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{P}_{q+1}\right), \phi\left(\vartheta\left(\mathfrak{P}_{q+1}\right)\right)\right)\right] = -\infty.$$

Again, by Definition 4, we get

$$\lim_{q\to\infty} J\left(\vartheta\left(\mathfrak{P}_{q+1}\right), \phi\left(\vartheta\left(\mathfrak{P}_{q+1}\right)\right)\right) = 0.$$

It gives $\lim_{q\to\infty} \vartheta(\mathfrak{P}_q) = 0 = \lim_{q\to\infty} \phi\left(\vartheta(\mathfrak{P}_q)\right)$.

Since $\mathfrak{P}_q \supseteq \mathfrak{P}_{q+1}$ in the pursuit of hypothesis, we came to conclude that $\mathfrak{P}_\infty = \bigcap_{q=1}^{\infty} \mathfrak{P}_q$ is nonempty, closed and convex subset of $\mathfrak{P}$ and $\mathfrak{P}_\infty$ is invariant under $\mathfrak{L}$. Thus Schauder's result implies that $\mathfrak{L}$ has a fixed point in $\mathfrak{P}_\infty \subseteq \mathfrak{P}$. This completes the proof. □

Theorem 4 *Let $\mathfrak{P}$ be a nonempty BCC subset of a Banach space $\mathfrak{H}$. Also $\mathfrak{L} : \mathfrak{P} \to \mathfrak{P}$ is continuous function such that*

$$\mathfrak{W}\left[\vartheta\left(\mathfrak{L}\mathcal{Q}\right) + \phi\left(\vartheta\left(\mathfrak{L}\mathcal{Q}\right)\right)\right] \leq \mathcal{V}\left\{\mathfrak{W}\left[\vartheta\left(\mathcal{Q}\right) + \phi\left(\vartheta\left(\mathcal{Q}\right)\right)\right]\right\} \tag{4}$$

for all $\mathcal{Q} \subseteq \mathfrak{P}$, $\mathfrak{W} \in \Delta$, $\mathcal{V} \in \Theta$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, where ϑ is an arbitrary MNC. Then $\mathfrak{L}$ has minimum of one fixed point in $\mathfrak{P}$.

Proof The result follows for $J(\iota, \varpi) = \iota + \varpi$ in Theorem 3. □

Theorem 5 *Let $\mathfrak{P}$ be a nonempty BCC subset of a Banach space $\mathfrak{H}$. Also $\mathfrak{L} : \mathfrak{P} \to \mathfrak{P}$ is a continuous function such that*

$$\tau + \mathfrak{W}\left[\vartheta\left(\mathfrak{L}\mathcal{Q}\right) + \phi\left(\vartheta\left(\mathfrak{L}\mathcal{Q}\right)\right)\right] \leq \mathfrak{W}\left[\vartheta\left(\mathcal{Q}\right) + \phi\left(\vartheta\left(\mathcal{Q}\right)\right)\right] \tag{5}$$

for all $\mathcal{Q} \subseteq \mathfrak{P}$, $\mathfrak{W} \in \Delta$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, where ϑ is an arbitrary MNC. Then $\mathfrak{L}$ has minimum of one fixed point in $\mathfrak{P}$.

Proof The result follows by taking $\mathcal{V}(t) = t - \tau,\ \tau > 0,\ t \in \mathbb{R}$ in Theorem 4. □

Theorem 6 *Let $\mathfrak{P}$ be a nonempty BCC subset of a Banach space $\mathfrak{H}$. Also $\mathfrak{L} : \mathfrak{P} \to \mathfrak{P}$ is a continuous function such that*

$$\vartheta\left(\mathfrak{L}\mathcal{Q}\right) + \phi\left(\vartheta\left(\mathfrak{L}\mathcal{Q}\right)\right) \leq k\left[\vartheta\left(\mathcal{Q}\right) + \phi\left(\vartheta\left(\mathcal{Q}\right)\right)\right] \tag{6}$$

for all $\mathcal{Q} \subseteq \mathfrak{P}$, $0 \leq k < 1$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, where ϑ is an arbitrary MNC. Then $\mathfrak{L}$ has minimum of one fixed point in $\mathfrak{P}$.

Proof The result follows for $\mathfrak{M}(t) = \ln(t),\ k = e^{-\tau} \in [0, 1)$ in Theorem 5. □

Remark 2 For $\phi \equiv 0$ in Theorem 6 we obtain Darbo's fixed point theorem.

Definition 6 (Chang and Huang 1996) An element $(p, q) \in \mathfrak{X} \times \mathfrak{X}$ is called a coupled fixed point of a mapping $\mathfrak{T} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ if $\mathfrak{T}(p, q) = p$ and $\mathfrak{T}(q, p) = q$.

Theorem 7 (Banaś and Goebel 1980) *Suppose $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ is the MNC in $E_1, E_2, \ldots, E_n$, respectively. Moreover, suppose the function $\mathfrak{X} : \mathbb{R}_+^n \to \mathbb{R}_+$ is convex and $\mathfrak{F}\left(p_1, p_2, \ldots, p_n\right) = 0$ if and only if $p_l = 0$ for $l = 1, 2, \ldots, n$ then $\vartheta(\mathfrak{X}) = \mathfrak{F}\left(\vartheta_1(\mathfrak{X}_1), \vartheta_2(\mathfrak{X}_2), \ldots, \vartheta_n(\mathfrak{X}_n)\right)$ define a MNC in $E_1 \times E_2 \times \ldots \times E_n$, where $\mathfrak{X}_l$ denotes the natural projection of $\mathfrak{X}$ into E_l for $l = 1, 2, \ldots, n$.*

Example 1 (Banaś and Goebel 1980) Let ϑ be a MNC on E. Define $\mathfrak{F}(p, q) = p + q,\ p, q \in \mathbb{R}_+$. Then $\mathfrak{F}$ has all the properties mentioned in Theorem 7. Hence, $\vartheta^{cf}(\mathfrak{X}) = \vartheta(\mathfrak{X}_1) + \vartheta(\mathfrak{X}_2)$ is a MNC in the space $E \times E$, where $\mathfrak{X}_l,\ l = 1, 2$ denote the natural projections of $\mathfrak{X}$.

Theorem 8 *Let $\mathfrak{P}$ be a nonempty BCC subset of a Banach space $\mathfrak{H}$. Also $\mathfrak{L} : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ is a continuous function such that*

$$\begin{aligned} &\mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right), \phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)\right)\right)\right] \\ &\quad \leq \frac{1}{2}\mathcal{V}\left\{\mathfrak{W}\left[J\left(\vartheta\left(\mathcal{Q}_1\right)+\vartheta\left(\mathcal{Q}_2\right), \phi\left(\vartheta\left(\mathcal{Q}_1\right)+\vartheta\left(\mathcal{Q}_2\right)\right)\right)\right]\right\} \end{aligned}$$

for all $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathfrak{P}$, where ϑ is an arbitrary MNC and $\mathfrak{W}$, $\mathcal{V}$, J and ϕ are as in Theorem 3. In addition we assume $\mathfrak{W}(p+q) \leq \mathfrak{W}(p)+\mathfrak{W}(q)$, $p, q \geq 0$ and $\phi(p+q) \leq \phi(p)+\phi(q)$, $p, q \geq 0$. Then $\mathfrak{L}$ has at least a coupled fixed point in $\mathfrak{P}$.

Proof Consider a mapping $\mathfrak{L}^{cf} : \mathfrak{P}\times\mathfrak{P} \to \mathfrak{P}\times\mathfrak{P}$ by $\mathfrak{L}^{cf}(x, y) = (\mathfrak{L}(x, y), \mathfrak{L}(y, x))$, x, y $s \in \mathfrak{P}$. It is trivial that $\mathfrak{L}^{cf}$ is continuous.

Let $\mathcal{Q} \subseteq \mathfrak{P} \times \mathfrak{P}$ be nonempty. We have $\vartheta^{cf}(\mathcal{Q}) = \vartheta(\mathcal{Q}_1) + \vartheta(\mathcal{Q}_2)$ is an NMC, where $\mathcal{Q}_1, \mathcal{Q}_2$ are the natural projections of $\mathcal{Q}$ into $\mathfrak{H}$.

We obtain

$$\begin{aligned} &\mathfrak{W}\left[J\left(\vartheta^{cf}\left(\mathfrak{L}^{cf}(\mathcal{Q})\right), \phi\left(\vartheta\left(\mathfrak{L}^{cf}(\mathcal{Q})\right)\right)\right)\right] \\ &\quad \leq \mathfrak{W}\left[J\left(\vartheta^{cf}\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right) \times \mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right), \phi\left(\vartheta^{cf}\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right) \times \mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right)\right)\right)\right] \\ &\quad = \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)+\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right), \phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)+\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right)\right)\right)\right] \\ &\quad \leq \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)+\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right), \phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)\right)+\phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right)\right)\right)\right] \\ &\quad \leq \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right), \phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)\right)\right)\right] \\ &\qquad + \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right), \phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_2 \times \mathcal{Q}_1\right)\right)\right)\right)\right] \\ &\quad \leq \mathcal{V}\left\{\mathfrak{W}\left[J\left(\vartheta\left(\mathcal{Q}_1\right)+\vartheta\left(\mathcal{Q}_2\right), \phi\left(\vartheta\left(\mathcal{Q}_1\right)+\vartheta\left(\mathcal{Q}_2\right)\right)\right)\right]\right\} \\ &\quad = \mathcal{V}\left\{\mathfrak{W}\left[J\left(\vartheta^{cf}(\mathcal{Q}), \phi\left(\vartheta^{cf}(\mathcal{Q})\right)\right)\right]\right\}. \end{aligned}$$

By Theorem 3 we conclude that $\mathfrak{L}^{cf}$ has minimum of one fixed point in $\mathfrak{P} \times \mathfrak{P}$, i.e., $\mathfrak{L}$ has minimum of one coupled fixed point. □

Corollary 1 *Let $\mathfrak{P}$ be a nonempty BCC subset of a Banach space $\mathfrak{H}$. Also $\mathfrak{L} : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ is a continuous function such that*

$$\begin{aligned} &\tau + \mathfrak{W}\left[J\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right), \phi\left(\vartheta\left(\mathfrak{L}\left(\mathcal{Q}_1 \times \mathcal{Q}_2\right)\right)\right)\right)\right] \\ &\quad \leq \mathfrak{W}\left[J\left(\vartheta\left(\mathcal{Q}_1\right)+\vartheta\left(\mathcal{Q}_2\right), \phi\left(\vartheta\left(\mathcal{Q}_1\right)+\vartheta\left(\mathcal{Q}_2\right)\right)\right)\right] \end{aligned}$$

for all $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathfrak{P}$, $\tau > 0$, where ϑ is an arbitrary MNC and $\mathfrak{W}$, J and ϕ are as in Theorem 3. Further, we assume $\mathfrak{W}(p+q) \leq \mathfrak{W}(p)+\mathfrak{W}(q)$, $p, q \geq 0$ and $\phi(p+q) \leq \phi(p)+\phi(q)$, $p, q \geq 0$. Then $\mathfrak{L}$ has at least a coupled fixed point in $\mathfrak{P}$.

Proof The result can be obtained by taking $\mathcal{V}(t) = t - 2\tau$, $\tau > 0$ in Theorem 8. □

3 Application

The fractional integral of a function $f \in L_1(a, b)$ by another function g of order α is defined by (see Samko et al. 1993)

$$I_{a,g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{(g(x)-g(t))^{1-\alpha}} dt, \ \alpha > 0, \ -\infty \leq a < b \leq \infty$$

defined for every continuous function $f(t)$ and for any monotone function $g(t)$ having a continuous derivative.

Recently in Nieto and Samet (2017), Nieto and Samet discussed the existence of solutions of the implicit integral equation

$$z(t)=\mathcal{F}\left(t, z(t), \phi\left(\int_a^t \frac{g'(s)h(t,s,z(s))}{(g(t)-g(s))^{1-\alpha}}ds\right)\right), \quad t\in[a,\tau], \tag{7}$$

where $\tau>0, a\geq 0, \alpha\in(0,1), \mathcal{F}:[a,\tau]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, \phi:\mathbb{R}\to\mathbb{R}, g:[a,\tau]\to\mathbb{R}$ and $h:[a,\tau]\times[a,\tau]\times\mathbb{R}\to\mathbb{R}$.

From the above Eq. (7), we are motivated to implement the generalized Darbo fixed point theorem and Hausdorff MNC for solvability of a system of implicit fractional integral equations in Banach space.

In this part, the existence of solution for the following system of implicit fractional integral equations will be studied

$$z_n(x)=\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x,w,z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right), \quad n\in\mathbb{N}, \tag{8}$$

where $0<\alpha<1,\ x\in I=[a,\tau],\ \tau>0,\ a\geq 0,\ z(x)=(z_n(x))_{n=1}^{\infty}\in\mathfrak{H}$ and $\mathfrak{H}$ is a Banach sequence space.

3.1 Existence of solution on $C(I, c_0)$

We consider the following assumptions:

(1) The functions $\mathcal{B}_n: I\times C(I,c_0)\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$|\mathcal{B}_n(x,z(x),l)-\mathcal{B}_n(x,\bar{z}(x),m)|\leq\alpha_n(x)\,|z_n(x)-\bar{z}_n(x)|+\beta_n(x)\,|l-m|$$

for $z(x)=(z_n(x))_{n=1}^{\infty},\ \bar{z}(x)=(\bar{z}_n(x))_{n=1}^{\infty}\in C(I,c_0)$ and $\alpha_n,\ \beta_n: I\to\mathbb{R}_+\ (n\in\mathbb{N})$ are continuous functions.
Also, $D_n=\sup\left\{\left|\mathcal{B}_n\left(x,z^0,0\right)\right|: x\in I\right\}$, where $z^0=\left(z_n^0(x)\right)_{n=1}^{\infty}\in C(I,c_0)$ such that $z_n^0(x)=0$ for all $x\in I,\ n\in\mathbb{N}$ and $\sup_{n\in\mathbb{N}}D_n=D,\ \lim_{n\to\infty}D_n=0$.

(2) The functions $H_n: I\times I\times C(I,c_0)\to\mathbb{R}(n\in\mathbb{N})$ are continuous and there exists

$$\hat{H}=\sup\{|H_n(x,w,z(w))|: x,w\in I; n\in\mathbb{N}; z(w)\in C(I,c_0)\}.$$

(3) The function $g: I\to\mathbb{R}_+$ is in C^1 and nondecreasing.

(4) Define an operator T from $I\times C(I,c_0)\times\mathbb{R}$ to $C(I,c_0)$ as follows

$$(x, z(x))\mapsto(Tz)(x),$$

where

$$(Tz)(x) = \left(\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x, w, z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right)\right)_{n=1}^{\infty}.$$

(5) Let

$$\sup_{x\in I}\alpha_n(x) = \hat{\alpha}_n,\ \sup_{x\in I}\beta_n(x) = \hat{\beta}_n,\ \sup_{n\in\mathbb{N}}\hat{\alpha}_n = \hat{\alpha},\ \sup_{n\in\mathbb{N}}\hat{\beta}_n = \hat{\beta}.$$

Also,

$$\lim_{n\to\infty}\hat{\alpha}_n = 0,\ \lim_{n\to\infty}\hat{\beta}_n = 0 \text{ and } 0 < \hat{\alpha} < 1.$$

Suppose $B = \left\{z \in C(I, c_0) : \| z \|_{C(I,c_0)} \le r\right\}$.

Theorem 9 *Under the hypothesis (1)–(5), Eq. (8) has minimum of one solution in $C(I, c_0)$.*

Proof For arbitrary $x \in I$,

$$\begin{aligned}
&\| z(x) \|_{c_0}\\
&= \sup_{n\ge1}\left|\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x, w, z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right)\right|\\
&\le \sup_{n\ge1}\left|\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x, w, z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right) - \mathcal{B}_n(x, z^0(x), 0)\right|\\
&\quad + \sup_{n\ge1}\left|\mathcal{B}_n(x, z^0(x), 0)\right|\\
&\le \sup_{n\ge1}\left[\alpha_n(x)\,|z_n(x)| + \beta_n(x)\left|\int_a^x \frac{g'(w)H_n(x, w, z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right| + D\right]\\
&\le \hat{\alpha}\,\| z(x) \|_{c_0} + \hat{\beta}\int_a^x \frac{g'(w)\,|H_n(x, w, z(w))|}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w + D\\
&\le \hat{\alpha}\,\| z(x) \|_{c_0} + \hat{\beta}\hat{H}\int_a^x \frac{g'(w)}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w + D\\
&\le \hat{\alpha}\,\| z(x) \|_{c_0} + \frac{\hat{\beta}\hat{H}}{\alpha}(g(\tau)-g(a))^{\alpha} + D.
\end{aligned}$$

Therefore,

$$(1-\hat{\alpha})\,\| z(x) \|_{c_0} \le \frac{\hat{\beta}\hat{H}}{\alpha}(g(\tau)-g(a))^{\alpha} + D$$

implies

$$\| z(x) \|_{c_0} \le \frac{\alpha D + \hat{\beta}\hat{H}\,(g(\tau)-g(a))^{\alpha}}{\alpha(1-\hat{\alpha})} = r(say).$$

Hence $\| z \|_{C(I,c_0)} \le r$.

Consider $T : I \times B \times \mathbb{R} \to B$ is an operator given by

$$(Tz)(x) = \left(\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x, w, z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right)\right)_{n=1}^{\infty} = \left((T_n z)(x)\right)_{n=1}^{\infty},$$

where $z(x) \in B,\ x \in I$.

By assumption (4) we have

$$\lim_{n\to\infty}(T_n z)(x) = 0.$$

Hence, $(Tz)(x) \in C(I, c_0)$.

Again $\| Tz \|_{C(I,c_0)} \leq r$, so T is self mapping on B.

Let $\bar{z}(x) = (\bar{z}_n(x))_{n=1}^{\infty} \in B$ and $\epsilon > 0$ be such that $\| z - \bar{z} \|_{C(I,c_0)} < \frac{\epsilon}{2\hat{\alpha}} = \delta$.

Again for arbitrary $x \in I$,

$$
\begin{aligned}
&|(T_n z)(x) - (T_n \bar{z})(x)| \\
&= \left| \mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w) H_n(x, w, z(w))}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w\right) - \mathcal{B}_n\left(x, \bar{z}(x), \int_a^x \frac{g'(w) H_n(x, w, \bar{z}(w))}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w\right) \right| \\
&\leq \alpha_n(x) |z_n(x) - \bar{z}_n(x)| + \beta_n(x) \int_a^x \frac{g'(w) |H_n(x, w, z(w)) - H_n(x, w, \bar{z}(w))|}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w.
\end{aligned}
$$

As functions H_n are continuous for all $n \in \mathbb{N}$ so for $\| z - \bar{z} \|_{C(I,c_0)} < \frac{\epsilon}{2\hat{\alpha}}$ for all $n \in \mathbb{N}$, we have

$$
|H_n(x, w, z(w)) - H_n(x, w, \bar{z}(w))| < \frac{\alpha \epsilon}{2\hat{\beta}(g(\tau) - g(a))^{\alpha}}.
$$

Therefore,

$$
\begin{aligned}
&|(T_n z)(x) - (T_n \bar{z})(x)| \\
&\leq \hat{\alpha}\, |z_n(x) - \bar{z}_n(x)| + \frac{\hat{\beta} \epsilon \alpha}{2\hat{\beta}(g(\tau) - g(a)^{\alpha}} \int_a^x \frac{g'(w)}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w \\
&\leq \hat{\alpha} \| z - \bar{z} \|_{C(I,c_0)} + \frac{\hat{\beta} \epsilon \alpha}{2\hat{\beta}(g(\tau) - g(a))^{\alpha}} \cdot \frac{(g(\tau) - g(a))^{\alpha}}{\alpha} \\
&< \epsilon.
\end{aligned}
$$

Thus, $\| Tz - T\bar{z} \|_{C(I,c_0)} < \epsilon$ when $\| z - \bar{z} \|_{C(I,c_0)} < \delta$; hence, T is continuous on B.

Finally,

$$
\begin{aligned}
&\chi_{c_0}(TB) \\
&= \lim_{n \to \infty} \sup_{z \in B} \max_{k \geq n} \left| \mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w) H_n(x, w, z(w))}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w\right) \right| \\
&\leq \lim_{n \to \infty} \sup_{z \in B} \max_{k \geq n} \left[\hat{\alpha}\, |z_k(x)| + \frac{\hat{\beta}_k \hat{H}\, (g(\tau) - g(a))^{\alpha}}{\alpha} \right],
\end{aligned}
$$

i.e.,

$$
\chi_{c_0}(TB) \leq \hat{\alpha} \chi_{c_0}(B).
$$

Therefore,

$$
\chi_{C(I,c_0)}(TB) \leq \hat{\alpha} \chi_{C(I,c_0)}(B).
$$

Thus, by assumption (5) and Remark 2 gives T has minimum of one fixed point in $B \subseteq C(I, c_0)$. Hence, equation (8) has minimum of one solution in $C(I, c_0)$. This completes the proof. □

Example 2

$$
z_n(x) = \frac{z_n(x)}{2n + x} + \frac{2}{n^2} \int_0^x \frac{w \cos(z_n(w))}{(x^2 - w^2)^{\frac{1}{2}} (w + n^2)} \mathrm{d}w, \tag{9}
$$

where $x \in I = [0, 1],\ n \in \mathbb{N}$.

Here $\mathcal{B}_n(x, z(x), l) = \frac{z_n(x)}{2n+x} + \frac{l}{n^2}$, $H_n(x, w, z(w)) = \frac{\cos(z_n(x))}{w+n^2}$, $g(x) = x^2$, $\alpha = \frac{1}{2}$, $a = 0$ and $T = 1$.

It is obvious that $\mathcal{B}_n$ is continuous for all $n \in \mathbb{N}$ and

$$\begin{aligned} &|\mathcal{B}_n(x, z(x), l) - \mathcal{B}_n(x, \bar{z}(x), m)| \\ &\le \frac{1}{2n+x}|z_n(x) - \bar{z}_n(x)| + \frac{1}{n^2}|l-m|. \end{aligned}$$

Also,

$$\begin{aligned} \alpha_n(x) &= \frac{1}{2n+x}, \hat{\alpha}_n = \frac{1}{2n}, \lim_{n\to\infty} \hat{\alpha}_n = 0, \hat{\alpha} = \frac{1}{2}, \\ \beta_n(x) &= \frac{1}{n^2}, \hat{\beta}_n = \frac{1}{n^2}, \lim_{n\to\infty} \hat{\beta}_n = 0, \hat{\beta} = 1, \\ D_n &= 0, D = 0, \lim_{n\to\infty} D_n = 0. \end{aligned}$$

The function $g(x) = x^2$ is in C^1 and nondecreasing. Again, the functions H_n are continuous for all $n \in \mathbb{N}$.

If $z \in C(I, c_0)$ then as $n \to \infty$ for all $x \in I$, we have

$$z_n(x) \to 0, \ \frac{2}{n^2}\int_0^x \frac{w\cos(z_n(w))}{\left(x^2-w^2\right)^{\frac{1}{2}}\left(w+n^2\right)}\mathrm{d}w \to 0.$$

Therefore, assumption (4) is satisfied. Thus all the assumptions of Theorem 9 are satisfied hence Eq. (9) has a solution in $C(I, c_0)$.

3.2 Existence of solution on $C(I, \ell_1)$

We consider the following assumptions:

(1) The functions $\mathcal{B}_n : I \times C(I, \ell_1) \times \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{R})$ are continuous and satisfies

$$|\mathcal{B}_n(x, z(x), l) - \mathcal{B}_n(x, \bar{z}(x), m)| \le \phi_n(x)|z_n(x) - \bar{z}_n(x)| + \psi_n(x)|l-m|$$

for $z(x) = (z_n(x))_{n=1}^{\infty}$, $\bar{z}(x) = (\bar{z}_n(x))_{n=1}^{\infty} \in C(I, \ell_1)$ and ϕ_n, $\psi_n : I \to \mathbb{R}_+$ $(n \in \mathbb{N})$ are continuous functions.

Also,

$$\sum_{n=1}^{\infty}\left|\mathcal{B}_n\left(x, z^0, 0\right)\right|$$

converges to zero for all $x \in I$, where $z^0 = \left(z_n^0(x)\right)_{n=1}^{\infty} \in C(I, \ell_1)$ such that $z_n^0(x) = 0$ for all $x \in I$, $n \in \mathbb{N}$.

(2) The functions $H_n : I \times I \times C(I, \ell_1) \to \mathbb{R}$ $(n \in \mathbb{N})$ are continuous and there exists

$$Q_k = \sup\{|H_k(x, w, z(w))| : x, w \in I; z(w) \in C(I, \ell_1)\},$$

where n, $k \in \mathbb{N}$. Also $\sup_{k\in\mathbb{N}} Q_k = \hat{Q}$.

(3) The function $g : I \to \mathbb{R}_+$ is in C^1 and nondecreasing.

(4) Define an operator T from $I \times C(I, \ell_1) \times \mathbb{R}$ to $C(I, \ell_1)$ as follows:

$$(x, z(x)) \mapsto (Tz)(x),$$

where

$$(Tz)(x) = \left(\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x,w,z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right)\right)_{n=1}^{\infty}.$$

(5) Let

$$\sup_{x\in I}\phi_n(x) = \hat{\phi}_n, \;\; \sup_{n\in\mathbb{N}}\hat{\phi}_n = \hat{\phi}, \; 0 < \hat{\phi} < 1.$$

Also, for all $x \in I$, the series $\sum_{n\geq 1}\psi_n(x)$ is convergent and

$$\sum_{n\geq 1}\psi_n(x) \leq \hat{\psi}.$$

Assume $B_1 = \left\{z \in C(I,\ell_1) : \| z \|_{C(I,\ell_1)} \leq \hat{r}\right\}$.

Theorem 10 *Under the hypothesis (1)–(5), Eq. (8) has minimum of one solution in* $C(I,\ell_1)$.

Proof For arbitrary fixed $x \in I$,

$$\begin{aligned}
&\| z(x) \|_{\ell_1} \\
&= \sum_{n\geq 1}\left|\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x,w,z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right)\right| \\
&\leq \sum_{n\geq 1}\left|\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x,w,z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right) - \mathcal{B}_n(x, z^0(x), 0)\right| + \sum_{n\geq 1}\left|\mathcal{B}_n(x, z^0(x), 0)\right| \\
&\leq \sum_{n\geq 1}\left[\phi_n(x)\,|z_n(x)| + \psi_n(x)\left|\int_a^x \frac{g'(w)H_n(x,w,z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right|\right] \\
&\leq \hat{\phi} \| z(x) \|_{\ell_1} + \sum_{n\geq 1}\psi_n(x)\int_a^x \frac{g'(w)\,|H_n(x,w,z(w))|}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w \\
&\leq \hat{\phi} \| z(x) \|_{\ell_1} + \sum_{n\geq 1}\psi_n(x)\int_a^x \frac{g'(w)\hat{Q}}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w \\
&\leq \hat{\phi} \| z(x) \|_{\ell_1} + \hat{\psi}\hat{Q}\int_a^x \frac{g'(w)}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w \\
&\leq \hat{\phi} \| z(x) \|_{\ell_1} + \frac{\hat{\psi}\hat{Q}}{\alpha}(g(\tau)-g(a))^{\alpha}.
\end{aligned}$$

Therefore,

$$(1-\hat{\phi}) \| z(x) \|_{\ell_1} \leq \frac{\hat{\psi}\hat{Q}}{\alpha}(g(\tau)-g(a))^{\alpha}$$

implies

$$\| z(x) \|_{\ell_1} \leq \frac{\hat{\psi}\hat{Q}\,(g(\tau)-g(a))^{\alpha}}{\alpha(1-\hat{\phi})} = r(say).$$

Hence, $\| z \|_{C(I,\ell_1)} \leq \hat{r}$.

Consider $T : I \times B_1 \to B_1$ be an operator given by

$$(Tz)(x) = \left(\mathcal{B}_n\left(x, z(x), \int_a^x \frac{g'(w)H_n(x,w,z(w))}{(g(x)-g(w))^{1-\alpha}}\mathrm{d}w\right)\right)_{n=1}^{\infty} = ((T_n z)(x))_{n=1}^{\infty},$$

where $z(x) \in B_1, \; x \in I$.

By assumption (4) it follows that

$$\sum_{n\geq 1}\left(T_n z\right)(x)$$

is finite and unique. Hence $(Tz)(x) \in C(I, \ell_1)$.

Again $\| Tz \|_{C(I,\ell_1)} \leq \hat{r}$, so T is self-mapping on B_1.

Let $\bar{z}(x) = (\bar{z}_n(x))_{n=1}^{\infty} \in B_1$ and $\epsilon > 0$ be such that $\| z - \bar{z} \|_{C(I,\ell_1)} < \frac{\epsilon}{2\hat{\phi}} = \delta$.

Again for arbitrary $x \in I$,

$$\begin{aligned}
&|(T_n z)(x) - (T_n \bar{z})(x)| \\
&= \left| \mathcal{B}_n \left(x, z(x), \int_a^x \frac{g'(w) H_n(x, w, z(w))}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w \right) - \mathcal{B}_n \left(x, \bar{z}(x), \int_a^x \frac{g'(w) H_n(x, w, \bar{z}(w))}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w \right) \right| \\
&\leq \phi_n(x) |z_n(x) - \bar{z}_n(x)| + \psi_n(x) \int_a^x \frac{g'(w) |H_n(x, w, z(w)) - H_n(x, w, \bar{z}(w))|}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w.
\end{aligned}$$

As functions H_n are continuous for all $n \in \mathbb{N}$ so for $\| z - \bar{z} \|_{C(I,\ell_1)} < \frac{\epsilon}{2\hat{\phi}}$ for all $n \in \mathbb{N}$, we have

$$|H_n(x, w, z(w)) - H_n(x, w, \bar{z}(w))| < \frac{\alpha\epsilon}{2\hat{\psi}(g(T) - g(a))^{\alpha}}.$$

Therefore,

$$\begin{aligned}
&\sum_{n\geq 1} |(T_n z)(x) - (T_n \bar{z})(x)| \\
&\quad \leq \hat{\phi} \sum_{n\geq 1} |z_n(x) - \bar{z}_n(x)| + \frac{\epsilon\alpha}{2\hat{\psi}(g(T) - g(a)^{\alpha}} \int_a^x \frac{g'(w) \sum_{n\geq 1} \psi_n(x)}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w \\
&\quad \leq \hat{\phi} \| z - \bar{z} \|_{C(I,\ell_1)} + \frac{\hat{\psi}\epsilon\alpha}{2\hat{\psi}(g(\tau) - g(a))^{\alpha}} . \frac{(g(T) - g(a))^{\alpha}}{\alpha} \\
&\quad < \epsilon.
\end{aligned}$$

Therefore, $\| Tz - T\bar{z} \|_{C(I,\ell_1)} < \epsilon$ when $\| z - \bar{z} \|_{C(I,\ell_1)} < \delta$ hence T is continuous on B_1.

Finally,

$$\begin{aligned}
&\chi_{\ell_1}(TB_1) \\
&\quad = \lim_{n\to\infty} \sup_{z\in B_1} \sum_{k\geq n} \left| \mathcal{B}_n \left(x, z(x), \int_a^x \frac{g'(w) H_n(x, w, z(w))}{(g(x) - g(w))^{1-\alpha}} \mathrm{d}w \right) \right| \\
&\quad \leq \lim_{n\to\infty} \sup_{z\in B_1} \left[\hat{\phi} \sum_{k\geq n} |z_k(x)| + \frac{\hat{Q}\,(g(\tau) - g(a))^{\alpha} \left(\sum_{k\geq n} \psi_k(x) \right)}{\alpha} \right],
\end{aligned}$$

i.e.,

$$\chi_{\ell_1}(TB_1) \leq \hat{\phi} \chi_{\ell_1}(B_1).$$

Therefore,

$$\chi_{C(I,\ell_1)}(TB_1) \leq \hat{\phi} \chi_{C(I,\ell_1)}(B_1).$$

Thus, assumption (5) and Remark 2 gives T a minimum of one fixed point in $B_1 \subseteq C(I, \ell_1)$. Hence, equation (8) has minimum of one solution in $C(I, \ell_1)$. This completes the proof. □

Example 3 We consider the following systems of fractional equations:

$$z_n(x) = \frac{z_n(x)}{3n^2 + x} + \frac{3}{n^2}\int_0^x \frac{w^2 \sin(z_n(w))}{(x^3 - w^3)^{\frac{1}{2}}(w + n^2)} dw, \tag{10}$$

where $x \in I = [0, 1],\ n \in \mathbb{N}$.

Here $\mathcal{B}_n(x, z(x), l) = \frac{z_n(x)}{3n^2 + x} + \frac{l}{n^2},\ H_n(x, w, z(w)) = \frac{\sin(z_n(x))}{w + n^2},\ g(x) = x^3,\ \alpha = \frac{1}{2},\ a = 0$ and $T = 1$.

It is obvious that $\mathcal{B}_n$ is continuous for all $n \in \mathbb{N}$ and

$$\begin{aligned} &|\mathcal{B}_n(x, z(x), l) - \mathcal{B}_n(x, \bar{z}(x), m)| \\ &\le \frac{1}{3n^2 + x} |z_n(x) - \bar{z}_n(x)| + \frac{1}{n^2} |l - m|. \end{aligned}$$

Also

$$\begin{aligned} &\phi_n(x) = \frac{1}{3n^2 + x}, \hat{\phi}_n = \frac{1}{3n^2}, \hat{\phi} = \frac{1}{3},\ \psi_n(x) = \frac{1}{n^2}, \\ &\sum_{n=1}^{\infty} \psi_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \sum_{n=1}^{\infty} \left|\mathcal{B}_n(x, z^0, 0)\right| = 0. \end{aligned}$$

The function $g(x) = x^3$ is in C^1 and nondecreasing. Again, the functions H_n are continuous for all $n \in \mathbb{N}$ and

$$|H_n(x, w, z(w))| \le \frac{1}{n^2},$$

which gives $Q_n = \frac{1}{n^2}$ and $\hat{Q} = 1$.

If $z \in C(I, \ell_1)$ then

$$\begin{aligned} &\sum_{n=1}^{\infty} \left| \frac{z_n(x)}{3n^2 + x} + \frac{3}{n^2}\int_0^x \frac{w^2 \sin(z_n(w))}{(x^3 - w^3)^{\frac{1}{2}}(w + n^2)} dw \right| \\ &\le \frac{1}{3 + x} \sum_{n=1}^{\infty} |z_n(x)| + 3\sum_{n=1}^{\infty} \int_0^x \frac{w^2 |\sin(z_n(w))|}{(x^3 - w^3)^{\frac{1}{2}}(w + n^2)} dw \\ &\le \frac{1}{3} \sum_{n=1}^{\infty} |z_n(x)| + 3\sum_{n=1}^{\infty} \int_0^x \frac{w^2 |z_n(w)|}{(x^3 - w^3)^{\frac{1}{2}}(w + n^2)} dw \\ &\le \frac{1}{3} \|z\|_{C(I,\ell_1)} + \left\{ \int_0^x \frac{3w^2}{(x^3 - w^3)^{\frac{1}{2}}(w + n^2)} dw \right\} \|z\|_{C(I,\ell_1)} \\ &\le \left(\frac{1}{3} + 2\right) \|z\|_{C(I,\ell_1)} \\ &= \frac{7}{3} \|z\|_{C(I,\ell_1)} < \infty. \end{aligned}$$

Therefore, assumption (4) is satisfied. Thus, all the assumptions of Theorem 10 are obtained. Hence, Eq. (10) has a solution in $C(I, \ell_1)$.

4 Conclusion

In this article, we introduced a new condensing operator and established a generalization of Darbo's fixed point theorem and finally apply it to a system of fractional integral equation by another function.

The main contribution of this article is that we have extended the results of the article (Mohammadi et al. 2020) to established a generalization of Darbo's fixed point and applied it to obtain the existence of solution of system of fractional integral equations by another function which is the generalization of many other fractional integrals. Hence, the method applied in this article can be applied to system of many other fractional integral equations which are just particular cases of fractional integral equation by another function.

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