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Solvability of fractional integral equations via Darbo's fixed point theorem

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Abstract
A fixed point theorem has been generalized herein, using a newly constructed contraction operator. In addition to this, the solvability of fractional integrals based on this generalized fixed point theorem along with suitable examples have also been reported in this article

Keywords Measure of noncompactness (MNC) · Fractional integral equations · Fixed point theorem

Mathematics Subject Classification 45G10 · 45P05 · 47H10

1 Introduction

Kuratowski [14] was the first to propose the concept of a MNC, which was key in the construction of the fixed point theory. G. Darbo [15] generalized the Schauder's fixed point theorem by incoroporating Kuratowski's MNC into his formulation. Following that, a number of authors analyzed and determined various problems in differential equations, integral equations, and integrodifferential equations by using these MNC in their studies.

The axiomatic definition of the MNC, which was defined by J. Banaś [7] in the year 1980, has gained much research interest for studying different problems.

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Darbo's theorem was generalized and applied by many researchers. Darbo's theorem is generalized yet again, this time with an application. MNC apply to solved for several forms of integral equations by many researchers and through of fractional order integral equations (see [3, 9, 11, 13, 16, 17, 21]). Das et al. [18] established some new generalizations using C-class functions. Parvaneh and Hussain [19] analyzed the concept of generalized Wardowski type fixed point theorems via α-admissible FG-contractions in b-metric spaces. Recently, the solvability of integral equations has been established (see [4–6, 10, 12, 23]).

In this work, we have presented Darbo's fixed point theorem to solve fractional integral equations in Banach space with the different functions are used. Herein, we also illustrated a suitable example for verifying the proposed theorem.

Let a real Banach space $(\mathfrak{Z}, \| \, . \, \|)$ and $B(\theta, r) = \{z \in \mathfrak{Z} : \| z - \theta \| \leq r\}$. If $\mathfrak{E}(\neq \phi) \subseteq \mathfrak{Z}$. Then

- $\bar{\mathfrak{E}} =$ the closure of $\mathfrak{E}$,
- $\bar{\mathfrak{E}} =$ the convex closure of $\mathfrak{E}$,
- $\mathfrak{M}_{\mathfrak{Z}} =$ collection of all non-empty and bounded subsets of $\mathfrak{Z}$,
- $\mathfrak{N}_{\mathfrak{Z}} =$ collection of all relatively compact sets,
- $\mathbb{R} = (-\infty, \infty)$, also
- $\mathbb{R}_+ = [0, \infty)$.

The definition of a MNC is as follows: [7].

Definition 1.1 A function $\Omega : \mathfrak{M}_{\mathfrak{Z}} \to [0, \infty)$ is said to be a MNC in $\mathfrak{Z}$ if it fulfills axioms:

(i) for all $\mathfrak{E} \in \mathfrak{M}_{\mathfrak{Z}}$, we have $\Omega(\mathfrak{E}) = 0$ gives $\mathfrak{E}$ is relatively compact.
(ii) $\ker \Omega = \left\{\mathfrak{E} \in \mathfrak{M}_{\mathfrak{Z}} : \Omega\left(\mathfrak{E}\right) = 0\right\} \neq \phi$ and $\ker \Omega \subset \mathfrak{N}_{\mathfrak{Z}}$.
(iii) $\mathfrak{E} \subseteq \mathfrak{E}_1 \implies \Omega\left(\mathfrak{E}\right) \leq \Omega\left(\mathfrak{E}_1\right)$.
(iv) $\Omega\left(\bar{\mathfrak{E}}\right) = \Omega\left(\mathfrak{E}\right)$.
(v) $\Omega\left(Conv\mathfrak{E}\right) = \Omega\left(\mathfrak{E}\right)$.
(vi) $\Omega\left(\chi\mathfrak{E} + (1-\chi)\,\mathfrak{E}_1\right) \leq \chi\Omega\left(\mathfrak{E}\right) + (1-\chi)\,\Omega\left(\mathfrak{E}_1\right)$ for $\chi \in [0, 1]$.
(vii) if $\mathfrak{E}_k \in \mathfrak{M}_{\mathfrak{Z}}$, $\mathfrak{E}_k = \bar{\mathfrak{E}}_k$, $\mathfrak{E}_{k+1} \subset \mathfrak{E}_k$ for $k = 1, 2, 3, 4, 5, \ldots$ and $\lim_{k\to\infty} \Omega\left(\mathfrak{E}_k\right) = 0$ then $\bigcap_{k=1}^{\infty} \mathfrak{E}_k \neq \phi$.

The *kernel of measure* Ω is defined to be the family $ker\Omega$. Since $\Omega(\mathfrak{E}_\infty) \leq \Omega(\mathfrak{E}_k)$, $\Omega(\mathfrak{E}_\infty) = 0$. So, $\mathfrak{E}_\infty = \bigcap_{k=1}^{\infty} \mathfrak{E}_k \in ker\Omega$.

Some important theorem and definition

The following are some fundamental theorems to recall:

Theorem 1.2 (Shauder [1]). Let $\mathfrak{U}$ be a non-empty, bounded, closed and convex subset(NBCCS) of a Banach Space $\mathfrak{Z}$. If $\mathfrak{G} : \mathfrak{U} \to \mathfrak{U}$ is a continuous and compact mapping, then it must have at least one fixed point.

Theorem 1.3 (Darbo[15]). Let $\mathfrak{U}$ be a NBCCS of a Banach Space $\mathfrak{Z}$. Assume $\mathfrak{G} : \mathfrak{U} \to \mathfrak{U}$ is a continuous mapping with a constant $\chi \in [0, 1$ such that

$$\Omega(\mathfrak{G}\mathfrak{B}) \leq \chi\Omega(\mathfrak{B}), \;\; \mathfrak{B} \subseteq \mathfrak{U}.$$

Then $\mathfrak{G}$ has a fixed point.

The following related concepts are needed to establish an extension of Darbo's fixed point theorem:

Definition 1.4 [2] A continuous function $\mathcal{D} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a function of $\mathcal{C}$- class if subsequent axioms hold true:

(1) $\mathcal{D}(q, x) \leq q$,
(2) $\mathcal{D}(q, x) = q$ implies that either $q = 0$ or $x = 0$. Also $\mathcal{D}(0, 0) = 0$. A $\mathcal{C}$- class function is symbolized by $\mathcal{C}$.

For example,

(1) $\mathcal{D}(q, x) = q - x$,
(2) $\mathcal{D}(q, x) = aq,\ 0 < a < 1$.

Definition 1.5 [8] Suppose that Δ is the set of all continuous maps $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ fulfilling the conditions given below:

(1) $\delta(x) = 0 \Longleftrightarrow x = 0$.
(2) δ is a non decreasing.
(3) $\delta(x) < x$, forall $x > 0$.

Definition 1.6 [2] Let Π denote the set of all continuous functions $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varpi(q) = 0$.

2 Fixed point theory

Theorem 2.1 *Let $\mathbb{F}$ be a NBCCS of a Banach space $\mathfrak{Z}$. Also $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ is continuous mapping with*

$$\delta[\varOmega\left(\mathcal{T}\mathfrak{E}\right)] \leq \mathcal{D}[\delta\left(\varOmega\left(\mathfrak{E}\right)\right), \varpi\left(\varOmega\left(\mathfrak{E}\right)\right)] \tag{2.1}$$

where $\mathfrak{E} \subset \mathbb{F}$ and $\varOmega$ is an arbitrary MNC and $\varpi \in \Pi,\ \delta \in \Delta$ also $\mathcal{D} \in \mathcal{C}$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{F}$.

Proof To get to our main finding, we first need to set up a nested sequence and then use properties of the measure of noncompactness. Let us form a sequence $\left\{\mathbb{F}_p\right\}_{p=1}^{\infty}$ with $\mathbb{F}_1 = \mathbb{F}$ and $\mathbb{F}_{p+1} = Conv(\mathcal{T}\mathbb{F}_p)$ for $p \in \mathbb{N}$. Also $\mathcal{T}\mathbb{F}_1 = \mathcal{T}\mathbb{F} \subseteq \mathbb{F} = \mathbb{F}_1,\ \mathbb{F}_2 = Conv(\mathcal{T}\mathbb{F}_1) \subseteq \mathbb{F} = \mathbb{F}_1$. By continuing in the similar manner gives $\mathbb{F}_1 \supseteq \mathbb{F}_2 \supseteq \mathbb{F}_3 \supseteq \ldots \supseteq \mathbb{F}_p \supseteq \mathbb{F}_{p+1} \supseteq \ldots$.

If there exists $p_0 \in \mathbb{N}$ satisfying $\mathcal{D}[\delta\left(\varOmega\left(\mathbb{F}_{p_0}\right)\right), \varpi\left(\varOmega\left(\mathbb{F}_{p_0}\right)\right)] = 0$. Then$\varOmega(\mathbb{F}_{p_0}) = 0$, so $\mathbb{F}_{p_0}$ is a compact set. In this case Schauder's theorem implies $\mathcal{T}$ has a fixed point in $\mathbb{F}$. Again, if $\mathcal{D}[\delta\left(\varOmega\left(\mathbb{F}_p\right)\right), \varpi\left(\varOmega\left(\mathbb{F}_p\right)\right)] > 0,\ \ p \in \mathbb{N}$. For $p \in \mathbb{N}$, we now have

$$\delta[\varOmega\left(\mathbb{F}_{p+1}\right)]$$

$$\begin{aligned}
&= \delta[\Omega\left(Conv\mathcal{T}\mathbb{F}_p\right)] \\
&= \delta[\Omega\left(\mathcal{T}\mathbb{F}_p\right)] \\
&\leq \mathcal{D}[\delta\left(\Omega\left(\mathbb{F}_p\right)\right), \varpi\left(\Omega\left(\mathbb{F}_p\right)\right)] \\
&\leq \delta\left(\Omega\left(\mathbb{F}_p\right)\right).
\end{aligned}$$

As δ is a non-decreasing mapping, we obtain

$$\Omega\left(\mathbb{F}_{p+1}\right) \leq \Omega\left(\mathbb{F}_p\right).$$

Then, $\left\{\Omega\left(\mathbb{F}_p\right)\right\}$ is a bounded below and decreasing. So, it is converges to $a = \inf\left\{\mathbb{F}_p\right\}$.

If possible let $a > 0$. By using (2.1), we get

$$\begin{aligned}
\delta\left(\mathbb{F}_{p+1}\right) &= \delta\left(Conv\mathcal{T}\mathbb{F}_p\right) \\
&\leq \mathcal{D}[\delta\left(\Omega\left(\mathbb{F}_p\right)\right), \varpi\left(\Omega\left(\mathbb{F}_p\right)\right)]
\end{aligned}$$

As $p \to \infty$, we obtain

$$\delta(a) \leq \mathcal{D}[\delta(a), \varpi(a)] \leq \delta(a).$$

This gives

$$\mathcal{D}[\delta(a), \varpi(a)] = \delta(a).$$

Using by (2) of Definition 1.4, we get

$$\delta(a) = 0 \ or \ \varpi(a) = 0.$$

Hence, $a = 0$.
i.e., $\lim_{p\to\infty} \Omega\left(\mathbb{F}_p\right) = 0$. Since $\mathbb{F}_p \supseteq \mathbb{F}_{p+1}$, by Definition 1.1, we obtain $\mathbb{F}_\infty = \bigcap_{p=1}^{\infty} \mathbb{F}_p$ is non-empty,closed and convex subset of $\mathbb{F}$ and $\mathbb{F}_\infty$ is $\mathcal{T}$ invariant. So, Theorem 1.2 concludes that $\mathcal{T}$ has a fixed point in $\mathbb{F}$. This completes the proof. □

Theorem 2.2 *Let $\mathbb{F}$ be a NBCCS of a Banach space $\mathfrak{Z}$. Also $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ is continuous mapping with*

$$\delta[\Omega\left(\mathcal{T}\mathfrak{E}\right)] \leq \delta\left(\Omega\left(\mathfrak{E}\right)\right) - \varpi\left(\Omega\left(\mathfrak{E}\right)\right)] \tag{2.2}$$

where $\mathfrak{E} \subset \mathbb{F}$ and Ω is an arbitrary MNC and $\delta \in \Delta$, $\varpi \in \Pi$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{F}$.

Proof Putting $\mathcal{D}(q, x) = q - x$ in Theorem 2.1 we get, the Theorem 2.2. □

Theorem 2.3 *Let $\mathbb{F}$ be a NBCCS of a Banach space $\mathfrak{Z}$. Also $\mathcal{T}:\mathbb{F}\to\mathbb{F}$ is continuous mapping with*

$$\delta[\Omega\,(\mathcal{T}\mathfrak{E})]\le a\delta\,(\Omega\,(\mathfrak{E})) \tag{2.3}$$

where $\mathfrak{E}\subset\mathbb{F}$ and Ω is an arbitrary MNC and $\delta\in\Delta$. Then $\mathcal{T}$ has at least one fixed point in $\mathbb{F}$.

Proof Putting $\mathcal{D}(q,x)=aq,\ 0<a<1$ in Theorem 2.1 we get, the Theorem 2.3. □

Corollary 2.4 *Let $\mathbb{F}$ be a NBCCS of a Banach space $\mathfrak{Z}$. Also $\mathcal{T}:\mathbb{F}\to\mathbb{F}$ is continuous mapping with*

$$\Omega\,(\mathcal{T}\mathfrak{E})\le a\Omega\,(\mathfrak{E}) \tag{2.4}$$

where $\mathfrak{E}\subset\mathbb{F}$ and Ω is an arbitrary MNC . Then $\mathcal{T}$ has at least one fixed point in $\mathbb{F}$.

Proof Setting $\delta(x)=x$ in Theorem 2.3 we get, Darbo's theorem. □

3 Measure of noncompactness on $C([0,I])$

Let $\mathfrak{Z}=C(U)$ be the space of real continuous functions on U, where $U=[0,I]$. So, equipped with

$$\|\ \Lambda\ \|=\sup\{|\Lambda(t)|:t\in U\},\ \ \Lambda\in\mathfrak{Z}.$$

Let $T(\neq\phi)\subseteq\mathfrak{Z}$ be bounded. For $\Lambda\in T$ and $\varepsilon>0$, denote by $\mu(\Lambda,\varepsilon)$ the modulus of the continuity of Λ, i.e.,

$$\mu(\Lambda,\varepsilon)=\sup\{|\Lambda(t_1)-\Lambda(t_2)|:t_1,t_2\in U,|t_1-t_2|\le\varepsilon\}.$$

Moreover, we set

$$\mu(T,\varepsilon)=\sup\{\mu(\Lambda,\varepsilon):\Lambda\in T\};\ \ \mu_0(T)=\lim_{\varepsilon\to 0}\mu(T,\varepsilon).$$

It is generally known that the function μ_0 is an MNC in $\mathfrak{Z}$, with $\Gamma(T)=\frac{1}{2}\mu_0(T)$ (see [7]) as the Hausdorff MNC Γ.

4 Solvability fractional integral equation

In this part, we show how our conclusions concerning the existence of a solution to a fractional integral equation in Banach space can be applied.

Consider the following fractional integral equation:

$$w(\varphi)=\Psi(\varphi,w(\varphi))+\frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{\sigma(\varphi,w(\vartheta))}{(\varphi-\vartheta)^{1-\omega}}d\vartheta, \tag{4.1}$$

where $0 \leq \omega < 1,\ \varphi \in U = [0, I]$.
Let

$$D_{e_0} = \{w \in \mathfrak{Z} : \| w \| \leq e_0\}.$$

Assume that

(A) $\Psi : U \times \mathbb{R} \to \mathbb{R}$ be continuous function and there exists constant $\beta_1 \geq 0$ satisfying

$$|\Psi(\varphi, w(\varphi)) - \Psi(\varphi, w_1(\varphi))| \leq \beta_1 |w(\varphi) - w_1(\varphi)|,\ \varphi \in U;\ w, w_1 \in \mathbb{R}$$

and

$$\hat{U} = \sup\{|\Psi(\varphi, 0)| : \varphi \in U\}.$$

(B) $\sigma : U \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous function and there exists a nondecreasing function $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$|\sigma(\varphi, w)| \leq \varpi(|w|);\ (\varphi, w) \in U \times \mathbb{R}.$$

(C) There exists a positive solution e_0 such that

$$\beta_1 e_0 + \hat{U} + \frac{\varpi(e_0)}{\Gamma(\omega + 1)}.I^{\omega} \leq e_0.$$

Theorem 4.1 *If conditions (A)-(C) satisfied, so the Eq. (4.1) has at least a solution in* $\mathfrak{Z}$.

Proof Consider the mapping $\mathcal{S} : \mathfrak{Z} \to \mathfrak{Z}$ as follows:

$$(\mathcal{S}w)(\varphi) = \Psi(\varphi, w(\varphi)) + \frac{1}{\Gamma(\omega)} \int_0^{\varphi} \frac{\sigma(\varphi, w(\vartheta))}{(\varphi - \vartheta)^{1-\omega}} d\vartheta,$$

Step 1: We show that $\mathcal{S}$ maps D_{e_0} into D_{e_0}. Let $w \in D_{e_0}$, We have

$$\begin{aligned}
&|(\mathcal{S}w)(\varphi)| \\
&\leq |\Psi(\varphi, w(\varphi))| + \left| \frac{1}{\Gamma(\omega)} \int_0^{\varphi} \frac{\sigma(\varphi, w(\vartheta))}{(\varphi - \vartheta)^{1-\omega}} d\vartheta \right| \\
&\leq |\Psi(\varphi, w(\varphi)) - \Psi(\varphi, 0)| + |\Psi(\varphi, 0)| + \left| \frac{1}{\Gamma(\omega)} \int_0^{\varphi} \frac{\sigma(\varphi, w(\vartheta))}{(\varphi - \vartheta)^{1-\omega}} d\vartheta \right| \\
&\leq \beta_1 |w(\varphi)| + \hat{U} + \left| \frac{1}{\Gamma(\omega)} \int_0^{\varphi} \frac{\sigma(\varphi, w(\vartheta))}{(\varphi - \vartheta)^{1-\omega}} d\vartheta \right|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{\sigma(\varphi, w(\vartheta))}{(\varphi-\vartheta)^{1-\omega}}d\vartheta\right| \\
&\leq \frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{|\sigma(\varphi, w(\vartheta))|}{(\varphi-\vartheta)^{1-\omega}}d\vartheta \\
&\leq \frac{\varpi(\| w \|)}{\Gamma(\omega)}\int_0^{\varphi}\frac{1}{(\varphi-\vartheta)^{1-\omega}}d\vartheta \\
&\leq \frac{\varpi(\| w \|)}{\Gamma(\omega+1)}I^{\omega}.
\end{aligned}$$

Hence, $\| w \| < e_0$ gives

$$\| \mathcal{S}w \| \leq \beta_1 e_0 + \hat{U} + \frac{\varpi(e_0)}{\Gamma(\omega+1)}I^{\omega} \leq e_0.$$

Due to the assumption (C) ,$\mathcal{S}$ maps D_{e_0} into D_{e_0}.

Step 2: We show that $\mathcal{S}$ is continuous on D_{e_0}. Let $\varepsilon > 0$ and $w, w_1 \in D_{e_0}$ such that $\| w - w_1 \| < \varepsilon$. For all $\varphi \in U$, we have

$$\begin{aligned}
&|(\mathcal{S}w)(\varphi) - (\mathcal{S}w_1)(\varphi)| \\
&\leq |\Psi(\varphi, w(\varphi)) - \Psi(\varphi, w_1(\varphi))| + \left|\frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{\sigma(\varphi, w(\vartheta))}{(\varphi-\vartheta)^{1-\omega}}d\vartheta\right. \\
&\quad \left. - \frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{\sigma(\varphi, w_1(\vartheta))}{(\varphi-\vartheta)^{1-\omega}}d\vartheta\right| \\
&\leq \beta_1 |w(\varphi) - w_1(\varphi)| + \frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{1}{(\varphi-\vartheta)^{1-\omega}}|\sigma(\varphi, w(\vartheta)) - \sigma(\varphi, w_1(\vartheta))|\, d\vartheta \\
&\leq \beta_1 \| w - w_1 \| + \frac{1}{\Gamma(\omega)}\int_0^{\varphi}\frac{1}{(\varphi-\vartheta)^{1-\omega}}|\sigma(\varphi, w(\vartheta)) - \sigma(\varphi, w_1(\vartheta))|\, d\vartheta \\
&< \beta_1 \| w - w_1 \| + \frac{1}{\Gamma(\omega)}\mu_{e_0}(\varepsilon)\int_0^{\varphi}\frac{1}{(\varphi-\vartheta)^{1-\omega}}d\vartheta \\
&< \beta_1 \| w - w_1 \| + \frac{1}{\Gamma(\omega+1)}\mu_{e_0}(\varepsilon)I^{\omega}.
\end{aligned}$$

where

$$\mu_{e_0}(\varepsilon) = \sup\left\{\begin{array}{c} |\sigma(\varphi, w) - \sigma(\varphi, w_1)| : |w - w_1| \leq \varepsilon; \varphi \in U; \\ w, w_1 \in [-e_0, e_0] \end{array}\right\}.$$

Hence, $\| w - w_1 \| < \varepsilon$ gives

$$|(\mathcal{S}w)(\varphi) - (\mathcal{S}w_1)(\varphi)| < \beta_1 \varepsilon + \frac{1}{\Gamma(\omega+1)}\mu_{e_0}(\varepsilon)I^{\omega}.$$

As $\varepsilon \to 0$, we get $|(\mathcal{S}w)(\varphi) - (\mathcal{S}w_1)(\varphi)| \to 0$.

This clearly prove that $\mathcal{S}$ is continuous on D_{e_0}.

Step 3: An estimation of $\mathcal{S}$ with respect to μ_0: Now, asusuming $\Delta(\neq \phi) \subseteq D_{e_0}$. Let $\varepsilon > 0$ be arbitrary and choose $w \in \Delta$ and $\varphi_1, \varphi_2 \in U$ such that $|\varphi_2 - \varphi_1| \leq \varepsilon$ and $\varphi_2 \geq \varphi_1$.

Now,

$$\begin{aligned}
&|(\mathcal{S}w)(\varphi_2) - (\mathcal{S}w)(\varphi_1)| \\
&= \left| \Psi(\varphi_2, w(\varphi_2)) + \frac{1}{\Gamma(\omega)} \int_0^{\varphi_2} \frac{\sigma(\varphi_2, w(\vartheta))}{(\varphi_2 - \vartheta)^{1-\omega}} d\vartheta - \Psi(\varphi_1, w(\varphi_1)) \right. \\
&\quad \left. - \frac{1}{\Gamma(\omega)} \int_0^{\varphi_1} \frac{\sigma(\varphi_1, w(\vartheta))}{(\varphi_1 - \vartheta)^{1-\omega}} d\vartheta \right| \\
&\leq |\Psi(\varphi_2, w(\varphi_2)) - \Psi(\varphi_1, w(\varphi_1))| \\
&\quad + \frac{1}{\Gamma(\omega)} \left(\int_0^{\varphi_2} \frac{|\sigma(\varphi_2, w(\vartheta))|}{(\varphi_2 - \vartheta)^{1-\omega}} d\vartheta - \int_0^{\varphi_1} \frac{|\sigma(\varphi_1, w(\vartheta))|}{(\varphi_1 - \vartheta)^{1-\omega}} d\vartheta \right) \\
&\leq |\Psi(\varphi_2, w(\varphi_2)) - \Psi(\varphi_2, w(\varphi_1))| + |\Psi(\varphi_2, w(\varphi_1)) - \Psi(\varphi_1, w(\varphi_1))| \\
&\quad + \frac{\varpi(|w|)}{\Gamma(\omega)} \left(\int_0^{\varphi_2} \frac{1}{(\varphi_2 - \vartheta)^{1-\omega}} d\vartheta - \int_0^{\varphi_1} \frac{1}{(\varphi_1 - \vartheta)^{1-\omega}} d\vartheta \right) \\
&\leq \beta_1 |w(\varphi_2) - w(\varphi_1)| + |\Psi(\varphi_2, w(\varphi_1)) - \Psi(\varphi_1, w(\varphi_1))| + \frac{\varpi(e_0)}{\Gamma(\omega+1)} [\varphi_2^{\omega} - \varphi_1^{\omega}] \\
&\leq \beta_1 \mu(w, \varepsilon) + \mu_{\Psi}(e_0, \varepsilon) + \frac{\varpi(e_0)}{\Gamma(\omega+1)} [\varphi_2^{\omega} - \varphi_1^{\omega}].
\end{aligned}$$

where

$$\mu_{\Psi}(e_0, \varepsilon) = \sup \left\{ \begin{array}{c} |\Psi(\varphi_2, w) - \Psi(\varphi_1, w)| : |\varphi_2 - \varphi_1| \leq \varepsilon; \varphi_1, \varphi_2 \in U; \\ |w| \leq e_0 \end{array} \right\},$$

and,

$$\mu(w, \varepsilon) = \sup \left\{ |w(\varphi_2) - w(\varphi_1)| \leq \varepsilon : |\varphi_2 - \varphi_1| \leq \varepsilon; \varphi_1, \varphi_2 \in U \right\}.$$

As $\varepsilon \to 0$, then $\varphi_2 \to \varphi_1$, we get

$$\lim_{\varepsilon \to 0} \frac{\varpi(e_0)}{\Gamma(\omega+1)} [\varphi_2^{\omega} - \varphi_1^{\omega}] \to 0.$$

Hence,

$$|(\mathcal{S}w)(\varphi_2) - (\mathcal{S}w)(\varphi_1)| \leq \beta_1 \mu(w, \varepsilon) + \mu_{\Psi}(e_0, \varepsilon).$$

i.e.,

$$\mu(\mathcal{S}w, \varepsilon) \leq \beta_1 \mu(w, \varepsilon) + \mu_{\Psi}(e_0, \varepsilon).$$

By the uniform continuity of Ψ on $U \times [-e_0, e_0]$ we have $\lim_{\varepsilon \to 0} \mu_\Psi(e_0, \varepsilon) \to 0$, as $\varepsilon \to 0$.

Taking $\sup_{w \in \Delta}$ and $\varepsilon \to 0$ we get,

$$\mu_0(\mathcal{S}\Delta) \leq \beta_1 \mu_0(\Delta).$$

Thus by Corollary 2.4, $\mathcal{S}$ has a fixed point in $\Delta \subseteq D_{e_0}$
That is the Eq. (4.1) has a solution in $\mathfrak{Z}$. □

Example 4.2 Consider the fractional integral equation as follows:

$$w(\varphi) = \frac{w}{9+\varphi^2} + \frac{1}{\Gamma(\frac{1}{2})} \int_0^\varphi \frac{\cos^{-1}(\frac{w^2(\vartheta)}{1-\varphi^2})}{(\varphi-\vartheta)^{\frac{1}{2}}} d\vartheta. \tag{4.2}$$

for $\varphi \in [0, 3] = U$, which is a particular case of Eq. (4.1)

Here

$$\Psi(\varphi, w(\varphi)) = \frac{w}{9+\varphi^2};$$
$$\omega = \frac{1}{2}$$

and

$$\sigma(\varphi, w(\vartheta)) = \cos^{-1}(\frac{w^2(\vartheta)}{1-\varphi^2}).$$

Also, It is trivial that Ψ is a continuous satisfying

$$|\Psi(\varphi, w(\varphi)) - \Psi(\varphi, w_1(\varphi))| \leq \frac{|w - w_1|}{9}$$

Therefore, $\beta_1 = \frac{1}{9}$.
If $\| w \| \leq e_0$ then

$$\hat{U} = \frac{e_0}{9}$$

and

$$|\sigma(\varphi, w)| \leq \left| w^2 \right|.$$

So,

$$\varpi(e_0) = e_0^2.$$

Puting these values in the inequality of assumption (C) we get,

$$\frac{1}{9}e_0 + \frac{e_0}{9} + \frac{e_0^2}{\Gamma(\frac{3}{2})}(3)^{\frac{1}{2}} \leq e_0$$
$$\implies \frac{e_0}{\Gamma(\frac{3}{2})}(3)^{\frac{1}{2}} \leq \frac{7}{9}$$
$$\implies e_0 \leq \frac{7\Gamma(\frac{3}{2})}{9(3)^{\frac{1}{2}}}.$$

However, assumption (C) is also fulfilled for $e_0 = \frac{7\Gamma(\frac{3}{2})}{9(3)^{\frac{1}{2}}}$.

We can see that all of Theorem 4.1's assumptions are achieved, from (A) to (C). Equation (4.2), according to Theorem 4.1, has a solution in $\mathfrak{Z} = C(U)$.

Example 4.3 Consider the fractional integral equation as follows:

$$w(\varphi) = \frac{w}{3+\varphi^4} + \frac{1}{\Gamma(\frac{1}{3})}\int_0^{\varphi} \frac{\ln(\frac{w^4(\vartheta)}{1-\varphi^4})}{(\varphi-\vartheta)^{\frac{2}{3}}}d\vartheta. \tag{4.3}$$

for $\varphi \in [0, 1] = U$, which is a particular case of Eq. (4.1)

Here

$$\Psi(\varphi, w(\varphi)) = \frac{w}{3+\varphi^4};$$
$$\omega = \frac{1}{3}$$

and

$$\sigma(\varphi, w(\vartheta)) = \ln(\frac{w^4(\vartheta)}{1-\varphi^4}).$$

Also, It is trivial that Ψ is a continuous satisfying

$$|\Psi(\varphi, w(\varphi)) - \Psi(\varphi, w_1(\varphi))| \leq \frac{|w-w_1|}{3}$$

Therefore, $\beta_1 = \frac{1}{3}$.

If $\| w \| \leq e_0$ then

$$\hat{U} = \frac{e_0}{3}$$

and

$$|\sigma(\varphi, w)| \leq \left|w^2\right|.$$

So,

$$\varpi(e_0) = e_0^2.$$

Puting these values in the inequality of assumption (C) we get,

$$\begin{aligned}
\frac{1}{3}e_0 + \frac{e_0}{3} + \frac{e_0^2}{\Gamma(\frac{4}{3})}(1)^{\frac{1}{3}} &\leq e_0 \\
\implies \frac{e_0}{\Gamma(\frac{4}{3})} &\leq \frac{1}{3} \\
\implies e_0 &\leq \frac{\Gamma(\frac{4}{3})}{3}.
\end{aligned}$$

However, assumption (C) is also fulfilled for $e_0 = \frac{\Gamma(\frac{4}{3})}{3}$.

We can see that all of Theorem 4.1's assumptions are achieved, from (A) to (C). Equation (4.3), according to Theorem 4.1, has a solution in $\mathfrak{Z} = C(U)$.

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