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Nonessential sum graph of an Artinian ring

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Abstract

Purpose – The authors study the interdisciplinary relation between graph and algebraic structure ring defining a new graph, namely "non-essential sum graph". The nonessential sum graph, denoted by NES(R), of a commutative ring R with unity is an undirected graph whose vertex set is the collection of all nonessential ideals of R and any two vertices are adjacent if and only if their sum is also a nonessential ideal of R.

Design/methodology/approach – The method is theoretical.

Findings – The authors obtain some properties of NES(R) related with connectedness, diameter, girth, completeness, cut vertex, r-partition and regular character. The clique number, independence number and domination number of NES(R) are also found.

Originality/value – The paper is original.

Keywords Nonessential ideal, Nonessential sum graph, Minimal ideal

Paper type Research paper

1. Introduction

The growth of interdisciplinary study of graph and algebra took place after the introduction of zero-divisor graph by Istvan Back [1]. Some of the interesting graphs are comaximal graph of commutative ring [2], intersection graph of ideals of rings [3], total graph of commutative ring [4], etc. In [5], Atani *et al.* introduced a graph associated to proper nonsmall ideals of a commutative ring, namely, small intersection graph. The small intersection graph of a ring R, denoted by $G(R)$, is an undirected graph with vertex set is the collection of all nonsmall proper ideals of R and any two distinct vertices are adjacent if and only if their intersection is not small in R. Taking this insight of small intersection graph of a ring, we, in this paper, define nonessential sum graph of an Artinian ring.

To continue this sequel, we are going to remember some definitions and notations from ring and graph. Let R be a commutative ring with unity. An ideal I of R is said to be essential in R if $I \cap J \neq 0$, whenever J is a nonzero ideal of R. The sum of all minimal ideals of R is known as socle of R, denoted by soc(R). We use min(R) to denote the collection of all minimal ideals of R. The ring R is said to be an Artinian ring if every descending chain of R terminates. In an Artinian ring, every ideal contains a minimal ideal.

Let G be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. G is said to be a null graph if $V(G) = \phi$ and that G is said to be empty if $E(G) = \phi$. We denote degree of $v \in V(G)$ by $\deg(v)$. If $\deg(v) = 1$, then v is called an end vertex. G is complete if any two vertices are adjacent. G is said to be r-regular if degree of each vertex of G is r. A walk in G is an alternating sequence of vertices and edges, $v_0x_1v_1 \ldots x_nv_n$ in which each edge x_i is $v_{i-1}v_i$. A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a circuit is a closed walk which all its vertices are distinct (except the first and last).

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The length of a circuit is the number of edges in the circuit. The length of the smallest circuit of G is called the girth of G, denoted by girth(G). G is connected if there is a path between every two distinct vertices. G is disconnected if it is is not connected. A vertex of the connected graph G is said to be a cut vertex if removal of it makes G disconnected. If x and y are two distinct vertices of G, then $d(x, y)$ is the length of the shortest path from x to y and if there is no such path then $d(x, y) = \infty$. The diameter of G is the maximum distance among distances between all pair of vertices of G, denoted by diam(G). G is said to be a bipartite graph if the vertex set of G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins V_1 and V_2. If $|V_1| = m$, $|V_2| = n$ and if every vertex of V_1 (or V_2) is adjacent to all vertices of V_2, then the bipartite graph is said to be complete and is denoted by $K_{m,n}$. If either m or n is equal to 1, then $K_{m,n}$ is said to be a star. An r-partite graph is a graph whose vertex set is partitioned into r subsets with no edge has both ends in any one subset. If each vertex of a partite subset is joined to every vertex that is not in that partite subset, then the r-partite graph is said to be complete. A complete subgraph of G is called a clique. The number of vertices in the largest clique of G is called the clique number of G, denoted by $\omega(G)$. The neighborhood $N(v)$ of a vertex v in G is the set of vertices which are adjacent to v. For each $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A set of vertices S in G is a dominating set, if $N[S] = V$. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. An independent set of G is a set of vertices of G such that no two vertices are adjacent in that vertex set. The independence number of G is the number of vertices in the largest independence set in G, denoted by $\alpha(G)$.

In this paper, we introduce nonessential sum graph of commutative ring with unity. Let R be a commutative ring with unity. The nonessential sum graph of R, denoted by NES(R), is an undirected graph with vertex set as the collection of all nonessential ideals of R and any two vertices A and B are adjacent if and only if $A \cap B$ is also a nonessential ideal of R. In this article, we are mainly interested in nonessential sum graph of Artinian ring.

Any undefined terminology can be obtained in [7–8, 15–20].

2. Connectedness of nonessential sum graph

In this section, we obtain some results related to connectedness, diameter, girth, completeness, cut vertex, partiteness and regular character. We start with a remark.

Remark 2.1. An ideal A is nonessential ideal in R if and only if $A \subseteq \text{soc}(R)$. If B is a nonessential ideal of R then every ideal which is contained in B is also a nonessential ideal of R. If m is a minimal ideal of R and if A and B are two ideals such that $m \subseteq A + B$, then $m \subseteq A$ or $m \subseteq B$.

Lemma 2.2. If $\min(R) = \{m_i\}_{i \in \lambda}$, where λ is an index set and μ is a finite subset of λ, then $\sum_{\mu} m_i$ is a nonessential ideal of R.

Proof. If possible suppose $K = \sum_{\mu} m_i$ is an essential ideal of R. Since each $m_j \neq (0)$, so $K \cap m_j \neq (0)$ for $j \notin \mu$, which implies that $m_j \subseteq K$. But it is a contradiction by Remark 2.1. Hence the lemma. □

From this onwards, R is an Artinian ring.

Theorem 2.3. NES(R) is a null graph if and only if R contains exactly one minimal ideal.

Proof. First consider that NES(R) is a null graph. On the contrary, assume that m_1 and m_2 are two distinct minimal ideals of R. So $m_1 \cap m_2 = 0$ and this provides that both m_1 and m_2 are nonessential ideals of R, a contradiction. Conversely, suppose that R has exactly one minimal ideal m, say. If m is the only nontrivial proper ideal of R, then obviously NES(R) is a null graph. If A is a nontrivial proper ideal of R with $A \neq m$, then it is easy to observe that A is essential in R. The proof is complete. □

Theorem 2.4. NES(R) is an empty graph if and only if R has exactly two minimal ideals, which are the only nonessential ideals of R.

Proof. Let NES(R) be an empty graph. Then by Theorem 2.3 $|\min(R| \neq 1$. If $|\min(R)| \geq 3$ and $m_1, m_2, m_3 \in \min(R)$, then m_1 and m_2 are adjacent by Lemma 2.2. Therefore, $|\min(R)| = 2$ and so we take $|\min(R)| = \{m_1, m_2\}$ with $m_1 \neq m_2$. Clearly m_1 and m_2 are nonessential. If I is any other nonessential ideal which is different from m_1 and m_2, then $m_i \subset I$ for $i = 1, 2$. This gives that I and m_i are adjacent, a contradiction. Thus m_1 and m_2 are the only nonessential ideals of R. For the other direction, we consider R has exactly two minimal ideals, which are the only nonessential ideals of R. Then $m_1 + m_2 = \mathrm{soc}(R)$ is essential. So, NES(R) is an empty graph. This completes the proof. □

Theorem 2.5. The following statements are equivalent:

(1) NES(R) is disconnected.

(2) $|\min(R)| = 2$.

(3) NES(R) $= G_1 \cup G_2$, where G_1 and G_2 are two disjoint complete subgraphs of NES(R).

Proof. $(i) \Rightarrow (ii)$ Suppose that NES(R) is disconnected. We consider G_1 and G_2 are two components of NES(R) and I, J be two ideals such that $I \in V(G_1)$ and $J \in V(G_2)$. Take the minimal ideals m_1 and m_2 with $m_1 \subseteq I$ and $m_2 \subseteq J$. If $m_1 = m_2$, then $I - m_1 - J$ is a path, a contradiction. This asserts that $m_1 \neq m_2$. Again, if $|\min(R)| \geq 3$, then $m_1 + m_2$ is nonessential in R. From this we get $I - m_1 - m_2 - J$ is a path, a contradiction. Therefore $|\min(R)| = 2$.

$(ii) \Rightarrow (iii)$ Assume that $|\min(R)| = 2$. Then we obtain $\mathrm{soc}(R) = m_1 + m_2$, where m_1 and m_2 are the minimal ideals of R. Let $G_i = \{I \subseteq R : m_i \subseteq I \text{ and } I \text{ is nonessential in } R\}$. Let I and J be two nonadjacent vertices in G_1, then $I + J$ is essential in R, which implies $\mathrm{soc}(R) \subseteq I + J$. Hence $m_2 \subseteq I$ or $m_2 \subseteq J$, a contradiction because in that case either I is essential or J is essential. So, G_1 is complete subgraph of NES(R). In the same way, G_2 is also a complete subgraph of NES(R). Suppose K and L are two adjacent vertices where $K \in V(G_1)$ and $L \in V(G_2)$. Since $\mathrm{soc}(R) = m_1 + m_2 \subseteq K + L$, so $K + L$ is essential, a contradiction. Thus NES(R) $= G_1 \cup G_2$, where G_1 and G_2 are two disjoint complete subgraphs of NES(R).

$(iii) \Rightarrow (i)$ The proof is obvious.

Theorem 2.6. The diameter of NES(R) is $1, 2$ or ∞.

Proof. If NES(R) is disconnected then $\mathrm{diam}(\mathrm{NES}(R)) = \infty$. Suppose that NES($R$) is connected. If I and J are two nonadjacent vertices of NES(R) then $I + J$ is essential in R. Consider the minimal ideals m_1 and m_2 with $m_1 \subseteq I$ and $m_2 \subseteq J$. If $m_1 + J$ is nonessential, then $I - m_1 - J$ is a path, which gives $d(I, J) = 2$. Similarly, if $m_2 + I$ is nonessential in R, then $d(I, J) = 2$. Suppose that $m_1 + J$ and $m_2 + I$ are both essential in R. Since NES(R) is connected, so $|\min(R)| \geq 3$. Let $m_3 \in \min(R)$. Since $I + J$ is essential in R, therefore $m_3 \subseteq I + J$. This implies $m_3 \subseteq I$ or $m_3 \subseteq J$. If we take $m_3 \subseteq I$ then obviously $m_3 + I$ is nonessential in R. We assert that $m_3 + J$ is nonessential. If possible, $m_3 + J$ is essential in R, then $m_1 \subseteq \mathrm{soc}(R) \subseteq m_3 + J$, which gives $m_1 \subseteq J$. Hence $m_1 + J = J$ is nonessential, a contradiction. Therefore $I - m_3 - J$ is a path. Thus $\mathrm{diam}(I, J) = 2$. □

Theorem 2.7. If NES(R) contains a cycle, then $\mathrm{girth}(\mathrm{NES}(R)) = 3$.

Proof. First if we consider $|\min(R)| = 2$, then by Theorem 2.5 NES(R) $= G_1 \cup G_2$, where G_1 and G_2 are two disjoint complete subgraphs of NES(R). Therefore in this case, $\mathrm{girth}(\mathrm{NES}(R)) = 3$, whenever NES($R$) contains a cycle. Next, when $|\min(R)| \geq 3$, $m_1 + m_2$, $m_2 + m_3$, $m_3 + m_1$ are nonessential in R where $m_i \in \min(R)$, $i = 1, 2, 3$. Thus $m_1 - m_2 - m_3 - m_1$ is a cycle. Hence $\mathrm{girth}(\mathrm{NES})(R) = 3$. □

Theorem 2.8. Let R contain finitely many minimal ideals, then the following holds:

(1) There exists no vertex in NES(R) which is adjacent to every other vertex.

(2) NES(R) is not a complete graph.

Proof. To prove (i), let $\min(R) = \{m_1, m_2, \ldots, m_t\}$. Assume that there exists a vertex I in NES(R) such that I is adjacent to every other vertex. Let $m_i \subseteq I$ for some i. Let $K = \sum_{j \neq i} m_j$, which is nonessential in R. Thus K is a vertex in NES(R). Now, $K + I \supseteq \sum_{j \neq i} + m_i = \mathrm{soc}(R)$. Hence $K + I$ is essential, a contradiction to the fact that I is adjacent to every other vertex. Hence the result.

(ii) Clearly NES(R) is not complete by (i). □

Theorem 2.9. If NES(R) is connected, then NES(R) has no cut vertex.

Proof. On the contrary assume that I is a cut vertex of NES(R). Then NES$(R)\backslash\{I\}$ is disconnected. Thus, there are vertices J and K with I lies in every path joining K to J. By Theorem 2.6, $d(K, J) = 2$ and therefore $J - I - K$ is a path. We claim that I is a minimal ideal of R. If not, there exists an ideal L of R such that $L \subseteq I$. As I is nonessential in R, therefore L is also nonessential in R. Since $J + L \subseteq J + I$ and $J + I$ is nonessential in R, so $J + L$ is nonessential in R. In the same direction, $K + L$ is also nonessential in R. So, $J - L - K$ is a path in NES$(R)\backslash\{I\}$, which is a contradiction. Thus, I is a minimal ideal of R. Now, we assert that there exist a minimal ideal $m_i \neq I$ of R such that $m_i \nsubseteq J$. If not then $m_i \subseteq J$ for each $I(\neq m_i) \in \min(R)$ and so $\sum_{m_i \neq I} m_i \subseteq J$. This gives that $\mathrm{soc}(R) = I + \sum_{m_i \neq I} m_i \subseteq I + J$, a contradiction to the fact that $I + J$ is nonessential. Similarly, there exists $m_j(\neq I)$ such that $m_j \nsubseteq K$. Now we see that for each $m_t \in \min(R)$ either $m_t \subseteq J$ or $m_t \subseteq K$. Since $J + K$ is essential, $m_t \subset soc(R) \subseteq J + K$, which implies $m_t \subseteq J$ or $m_t \subseteq K$. Let $I \neq m_i$, $m_j \in \min(R)$ such that $m_i \nsubseteq J$ and $m_j \nsubseteq K$. Therefore, $m_i \subseteq K$ and $m_j \subseteq J$. So, $K - m_i - m_j - J$ is a path in NES$(R)\backslash\{I\}$, a contradiction. Therefore, NES(R) has no cut vertex. □

Theorem 2.10. NES(R) is not a complete r-partite graph.

Proof. If possible assume that NES(R) is a complete r-partite graph with r parts $V_1, V_2, \ldots, V_r$. Since two minimal ideals are always adjacent, by Remark 2.1, so each V_i contains at most one minimal ideal. Thus we get $|\min(R)| \leq r$. Our claim is $|\min(R)| = r$. Suppose $\min(R) = \{m_1, m_2, \ldots, m_t\}$ and $t < r$. Without loss of generality we can take $m_i \in V_i$ for $1 \leq i \leq t$. So, V_{t+1} contains no minimal ideal. Since $\min(R)$ is finite, so $\sum_{j \neq i} m_j$ is nonessential in R. Now, $\sum_{j \neq i} m_j + m_i = \mathrm{soc}(R)$, so $\sum_{j \neq i} m_j$ and m_i are not adjacent. Thus $\sum_{j \neq i} m_j \in V_i$ as $m_i \in V_i$. Let $I \in V_{t+1}$ and $m_k \subseteq I$ for some $m_k \in \min(R)$. So, I is adjacent to m_k. Since NES(R) is assumed to be complete r-partite and $m_k \in V_k$, so I is adjacent to every element of V_k, which implies I is adjacent to $\sum_{i \neq k} m_i$, a contradiction. Therefore, $|\min(R)| = r$. Now, consider $J = \sum_{i=3}^{r} m_i$. Clearly J is nonessential in R by Remark 2.1. As J is adjacent to m_1 and m_2, so $J \notin V_1, V_2$. Moreover, $J + m_i = J$ for $3 \leq i \leq r$. So, J is adjacent to all minimal ideals of R. We get that $J \notin V_i$ for each i, a contradiction. Hence the theorem. □

Theorem 2.11. The following statements holds:

(1) NES(R) contains an end vertex if and only if NES$(R) = G_1 \cup G_2$, where G_1 and G_2 are two disjoint complete subgraphs of NES(R) and $|V(G_i)| = 2$ for some $i = 1, 2$.

(2) NES(R) is not a star graph.

Proof. (i) Let I be an end vertex of NES(R). So, $\deg(I) = 1$. Suppose $|\min(R)| \geq 3$. For each $m_i \in \min(R)$, m_i is adjacent to every other minimal ideal of R, so $\deg(m_i) \geq 2$. Hence I is not a minimal ideal. We can assume $m_1 \subseteq I$. Hence I and m_1 are adjacent. Since $\deg(I) = 1$, so the only vertex adjacent to I is m_1 and $m_j \nsubseteq I$, $j \neq 1$. Again I and m_2 are not adjacent, so $I + m_2$ is essential. So we get, $m_j \subseteq \mathrm{soc}(R) \subseteq I + m_2$ for $j \neq 1, 2$, which implies $m_j \subseteq I$ for $j \neq 1$, a contradiction. So, $|\min(R)| = 2$. By Theorem 2.5, $\mathrm{NES}(R) = G_1 \cup G_2$, where G_1 and G_2 are two disjoint complete subgraphs of NES(R). Let $I \in V(G_i)$. Since G_i is a complete subgraph and $\deg(I) = 1$, so $|V(G_i)| = 2$. The converse part is clear.

(ii) Suppose that NES(R) is a star graph. So, NES(R) contains an end vertex. By the previous part $|\min(R)| = 2$ and then by Theorem 2.5, the graph is disconnected. Hence, NES(R) is not a star graph. □

Theorem 2.12. The following statements holds:

(1) If I and J are two vertices of NES(R) such that $I \subseteq J$, then $\deg(I) \geq \deg(J)$.

(2) If NES(R) is an r-regular graph then $|V(\mathrm{NES}(R))| = 2r + 2$.

Proof. (i) Suppose I and J are two vertices of NES(R) such that $I \subseteq J$. Let K be a vertex adjacent to J. So, $J + K$ is nonessential in R. As $I + K \subseteq J + K$, so $I + K$ is nonessential in R. Thus, each vertex adjacent to J is also adjacent to I. Hence $\deg(I) \geq \deg(J)$.

(ii) Let NES(R) be an r-regular graph. So, for each $m_i \in \min(R)$, $\deg(m_i) = r$. Since m_i is adjacent to each minimal ideal, by Remark 2.1, so $\min(R)$ is finite. Suppose, $|\min(R)| \geq 3$, so $\deg(m_1 + m_2) \leq \deg(m_1)$ by (i). Also, $\deg(m_1 + m_2) \neq \deg(m_1)$, since $\sum_{j \neq 2} m_j$ is adjacent to m_1 but not to $m_1 + m_2$. Thus, $\deg(m_1 + m_2) < \deg(m_1)$, a contradiction. So, $|\min(R)| \leq 2$. If $|\min(R)| = 1$ then NES(R) is null. Therefore, $|\min(R)| = 2$. By Theorem 2.5, $\mathrm{NES}(R) = G_1 \cup G_2$, where G_1 and G_2 are two disjoint complete subgraphs of NES(R). Let $\min(R) = \{m_1, m_2\}$ and $m_1 \in G_1$. Since $\deg(m_1) = r$, so $|G_1| = r + 1$. In the same direction, $|G_2| = r + 1$. Hence, $|V(\mathrm{NES}(R))| = 2r + 2$. □

3. Clique number, independence number, domination number of nonessential sum graph

In this section, we will find clique number, independence number, domination number of NES(R).

Theorem 3.1. The following holds:

(1) $\omega(\mathrm{NES}(R)) \geq |\min(R)|$.

(2) If $\omega(\mathrm{NES}(R)) < \infty$, then number of minimal ideals of R is finite.

(3) $\omega(\mathrm{NES}(R)) = 1$ if and only if $\min(R) = \{m_1, m_2\}$ and these two are the only nonessential ideals in R.

(4) If the number of minimal ideals of R is finite, then $\omega(\mathrm{NES}(R)) \geq 2^{|\min(R)|-1} - 1$.

Proof. (i) Since any two minimal ideals of R are adjacent, by Lemma 2.2, the subgraph with vertex set $\{m_i\}_{m_i \in \min(R)}$ of NES(R) is complete. So, $\omega(\mathrm{NES}(R)) \geq |\min(R)|$.

(ii) If $\omega(\mathrm{NES}(R)) < \infty$, then by (i) the number of minimal ideals of R is finite.

(iii) It is clear from Theorem 2.4.

(iv) Let $\min(R) = \{m_1, m_2, \ldots, m_t\}$ and for each $1 \leq i \leq t$, take $A_i = \{m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_t\}$. Let $P(A_i)$ be the power set of A_i. For each $X (\neq \{\}) \in P(A_i)$, consider $R_X = \sum_{T \in X} T$. Clearly T_X is nonessential. Also, subgraph with vertex set $\{R_X\}_{X \in P(A_i)}$ is a complete subgraph which is clear by Lemma 2.2. Now, $|P(A_i) \setminus \{\}| = 2^{|\min(R)|-1} - 1$. Therefore $|\{R_X\}_{X \in P(A_i)}| = 2^{|\min(R)|-1} - 1$. Hence, $\omega(\mathrm{NES}(R)) \geq 2^{|\min(R)|-1} - 1$. □

Theorem 3.2. The following holds:

(1) $\gamma(\mathrm{NES}(R)) \le 2$.

(2) $\min(R)$ is finite if and only if $\gamma(\mathrm{NES}(R)) = 2$ and $\min(R)$ is infinite if and only if $\gamma(\mathrm{NES}(R)) = 1$.

Proof. (i) Since NES(R) is not a null graph, $|\min(R)| \ge 2$. Consider $T = \{m_1, m_2\}$, where $m_1, m_2 \in \min(R)$. Take a vertex I in NES(R). If $m_1 \subseteq I$ or $m_2 \subseteq I$, then $m_1 + I$ or $m_2 + I$ is non-essential in R. Then I is adjacent to m_1 or m_2. Suppose that $m \not\subseteq I$ and $m \not\subseteq J$. If I is not adjacent to m_1, then $m_1 + I$ is essential in R. So, $m_2 \subseteq \mathrm{soc}(R) \subseteq m_1 + I$, which implies $m_2 \subseteq I$, a contradiction. Therefore I is adjacent to m_1. In the same way, I is adjacent to m_2. Thus $\gamma(\mathrm{NES}(R)) \le 2$.

(ii) If $\min(R)$ is finite, then by Theorem 2.8, there exists no vertex which is adjacent to every other vertex. So, $\gamma(\mathrm{NES}(R)) \ne 1$. Therefore, $\gamma(\mathrm{NES}(R)) = 2$ by part (i). In the opposite direction, let $\gamma(\mathrm{NES}(R)) = 1$. So, the graph has a vertex which is adjacent to every other vertex. So the graph does not contain finite minimal ideals. Hence the result. □

Theorem 3.3. Let R contain finite number of minimal ideals. Then $\alpha(\mathrm{NES}(R)) = |\min(R)|$.

Proof. Let $\min(R)$ be finite and $\min(R) = \{m_1, m_2, \ldots, m_t\}$. Since $\{\sum_{j=1, j\ne 1}^{t} m_j\}_{i=1}^{t}$ is an independent set in NES(R), therefore $t \le \alpha(\mathrm{NES}(R))$. Assume that $\alpha(\mathrm{NES}(R))$ is equal to p and $S = \{I_1, I_2, \ldots, I_p\}$ is the maximal independent set. For each $I \in S$, I is nonessential in R. So, there exists a minimal ideal m such that $m \not\subseteq I$. If $p > t$, then there exists $1 \le i, j \le p$ and $m \in \min(R)$ such that $m \not\subseteq I_i$ and $m \not\subseteq I_j$. Thus $m \not\subseteq I_i + I_j$. Otherwise, $m \subseteq I_i + I_j$ which leads to a contradiction. As S is independent, so $I_i + I_j$ is essential, which implies $m \subseteq \mathrm{soc}(R) \subseteq I_i + I_j$, a contradiction. Therefore, $\alpha(\mathrm{NES}(R)) = |\min(R)|$. If we take $\alpha(\mathrm{NES}(R)) = \infty$, then by similar argument we get a contradiction. Hence, $\alpha(\mathrm{NES}(R)) = |\min(R)|$. □

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