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On total directed graphs of non-commutative rings

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Abstract

For a non-commutative ring R, the left total directed graph of R is a directed graph with vertex set as R and for the vertices x and y, x is adjacent to y if and only if there is a non-zero $r \in R$ which is different from x and y, such that $rx + yr$ is a left zero-divisor of R. In this paper, we discuss some very basic results of left (as well as right) total directed graph of R. We also study the coloring of left total directed graph of R directed graph.

Keywords: Non-commutative ring; Zero-divisor; Directed graph; Total graph; Clique

1. Introduction

The total graph of a commutative ring with unity was introduced by Anderson and Badawi in [1]. They considered the total graph $T(\Gamma(R))$ of a commutative ring R as an undirected graph with vertex set as R and any two vertices of $T(\Gamma(R))$ are adjacent if and only if their ring sum is a zero-divisor of R. They studied the characteristics of total graph and its two induced subgraphs by taking the cases when the set of zero-divisors of R is an ideal of R and when this set is not an ideal of R. In [2], Akbari et al. continued this concept of total graph. Ahmad Abbasi and Shokoofe Habibi [3] investigated the total graph of a commutative ring with respect to proper ideals. Anderson and Badawi [4] studied the total graph of a commutative ring without zero element. M. H. Shekarriz et al. observed some basic graph theoretic properties of the total graph of a finite commutative ring in [5]. The insight for total graph is extended to modules also. The total graph of a commutative ring with respect to the proper submodules of a module was interpreted by A. Abbasi and S. Habibi in [6]. The total torsion element graph of a module over a commutative ring was discussed by S. Atani and S. Habibi in [7]. These kinds of continuous extension of Anderson and Badawi's [1] work signify its utility in graphical aspects of ring-theoretic structures.

To interpret this concept in directed graph, we take the definition in a different way. We introduce left and right total directed graphs of non-commutative rings. The left total directed graph of a non-commutative ring R, denoted

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by $T_l(\Gamma(R))$, is a directed graph with all elements of R as vertices. Any $x, y \in R$, the vertex x is adjacent to y if and only if there exists a non-zero r, which is not equal to x and y, in R such that $rx + yr$ is a left zero-divisor of R. More specifically, the non-zero element r is considered either a (left) zero-divisor or a (left) regular element. In this paper, we take the non-zero element r as left non-zero zero-divisor for the adjacency relation of two vertices. Henceforth, we continue this discussion by taking the non-zero element r as left non-zero zero-divisor. The induced subgraphs $Z_l(\Gamma(R))$ and $Reg_l(\Gamma(R))$ of $T_l(\Gamma(R))$ with vertex sets $Z_l(R)$ (set of left zero-divisors of R) and $Reg_l(R)$ (set of left regular elements of R) respectively, are studied. The idea of its dual concept that is, right total directed graph $T_r(\Gamma(R))$ and its respective induced subgraphs $Z_r(\Gamma(R))$ and $Reg_r(\Gamma(R))$ of a non-commutative ring R, are also taken under consideration. We do not state or establish each and every dual concept in our discussion, as almost all results hold for the respective dual concept.

Note that the non-zero zero-divisor r is not independent for the choice of adjacency of two vertices. That is the underlying (undirected) graph of the directed graph is not simple. Recollect a simple (undirected) graph contains at most one edge (arc is called an edge in an undirected graph) between a pair of vertices. Though a part of our discussions is on basic characteristics of ring-theoretic concepts, yet this motivates us for the establishment of interesting results.

2. Preliminaries

Throughout our discussion, unless otherwise specified, rings mean non-commutative rings. From this onward, R denotes a non-commutative ring, G is a directed graph, and for any $A \subseteq R$, A^* contains all non-zero elements of A.

A digraph or directed graph G is a non-empty set of vertices, denoted by $V(G)$, and a collection of ordered pairs of distinct vertices. Any such pair (u, v) is called an arc or directed line and will usually be denoted uv or u adj v. If uv and vu are not arcs in G, then we say that u and v are not adjacent. If $a = uv$ is an arc of G, a is said to be incident out of u and incident into v. The number of arcs incident out of a vertex v is the out-degree of v and the number of arcs incident into v is its in-degree. A vertex v of G is called a sink, if the in-degree of v is positive and the out-degree of v is zero. The dual concept of sink is called source. G is said to be symmetric whenever u adj v, v adj u for the vertices u and v of G.

A walk $v_0x_1v_1 \cdots x_nv_n$, in a directed graph is an alternating sequence of vertices and arcs, in which each arc x_i is $v_{i-1}v_i$. The length of such a walk is n, the number of occurrences of arcs in it. A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a cycle or circuit is a closed walk with all vertices distinct (except the first and last). If there is a path from a vertex u to a vertex v, then v is said to be reachable from u. A digraph is strongly connected, if every two vertices are mutually reachable. G is said to be totally disconnected, if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of any shortest path from x to y. The diameter $diam(G)$ of a graph G is defined as $max\{d(u, v) : u, v \in V(G)\}$. We say that two (induced) subgraphs G_1 and G_2 of G are disjoint if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_1 (respectively, G_2). A digraph is a tournament if its underlying graph is a complete graph i.e. any two vertices of the underlying graph are adjacent. It must be noted that this underlying graph is simple. Since the non-zero element r is not independent in adjacency of two vertices in total directed graph, so an underlying graph of the respective directed subgraph may not be simple. Thus for the results of tournament of this paper, we consider those rings whose underlying graph of corresponding (left) total directed graph is simple.

The coloring of a directed graph is an assignment of colors to the vertices of the graph with no two adjacent vertices are of same color, i.e. if *u adj v* for the vertices u and v of a directed graph G, then u and v are of different colors. A clique of a directed graph is a maximal tournament subgraph. The minimum number of colors needed for proper coloring of a directed graph G is known as chromatic number of G, and is denoted by $\chi(G)$.

Next, we remember some definitions from ring-theoretic concepts. An element e_l of R is called left identity if $e_lx = x$, for all $x \in R$. The collection of all left identity elements of R is denoted by $E_l(R)$. The dual concept of left identity element is right identity element, and the set of all right identity elements of R is denoted by $E_r(R)$. An identity element of R is an element which is both a left identity element and a right identity element. If a ring has an identity element (unity), then it is unique. An element x of R is said to be a left inverse of an element y if $xy = e$, where e is unity of R. In the same way, right inverse can be defined. An invertible element of R is an element which has a left and right inverse such that both are equal. An element $a \in R$ is said to be left zero-divisor if there exists a non-zero element $b \in R$ such that $ab = 0$ in R. Right zero-divisors are defined similarly. The collection of left and right zero-divisors of R are denoted by $Z_l(R)$ and $Z_r(R)$ respectively. The elements, which are not left and right zero-divisors of R, are known as left and right regular elements of R, denoted by $Reg_l(R)$ and $Reg_r(R)$, respectively. An element x in R is said to be finite provided Rx (or xR) is a finite set.

Any undefined terminology can be obtained in [8–13].

The left total directed graph $T_l(\Gamma(R))$ of R is a directed graph with R as vertex set. Any two vertices x and y of $T_l(\Gamma(R))$, x is adjacent to y ($x\ adj\ y$) if and only if there exists a non-zero left zero-divisor r, which is distinct from x and y, in $Z_l(R)$ with $rx + yr \in Z_l(R)$. Similarly, for the right total directed graph $T_r(\Gamma(R))$, any two vertices x and y, x is adjacent to y ($x\ \overline{adj}\ y$) if and only if there exists a non-zero right zero-divisor r, which is distinct from x and y, in $Z_r(R)$ with $rx + yr \in Z_r(R)$. $Z_l(\Gamma(R))$ and $Reg_l(\Gamma(R))$ are two induced subgraphs of $T_l(\Gamma(R))$ with vertex sets $Z_l(R)$ and $Reg_l(R)$ respectively. In the same sense, $Z_r(\Gamma(R))$ and $Reg_r(\Gamma(R))$ are two induced subgraphs of $T_r(\Gamma(R))$ with vertex sets $Z_r(R)$ and $Reg_r(R)$ respectively.

One can see the independence of the non-zero zero-divisor r in the following two examples in the respective left and right adjacency relation.

Example 1. *Consider the ring* $R = \{(a_{ij})_{2\times 2} : a_{11}, a_{12} \in \mathbb{Z}_2, a_{21} = 0 = a_{22}\} = \{A_1, A_2, A_3, A_4\}$, *where* $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Example 2. *Consider the ring* $R = \{0, a, b, c\}$ *with addition and multiplication operations defined as follows:*

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

3. Total directed graph of R

In this section, we mainly discuss some basic properties of left total directed graph $T_l(\Gamma(R))$ of R and its induced subgraphs $Z_l(\Gamma(R))$ and $Reg_l(\Gamma(R))$. We interpret the very elementary concepts, left and right, with identity elements, invertible elements, zero-divisors and regular elements of R. In [14], Wu explored some these types of ring-theoretic concepts' correspondence with directed zero-divisor graphs of finite rings. It motivates us to approach the same in our discussion.

The first result of this section gives the idea of connectedness in total directed graph. We observe that $Z_l(\Gamma(R))$ is strongly connected for $|Z_l(R)| \geq 3$.

Theorem 3.1. *If* $|Z_l(R)| \geq 3$, *then* $Z_l(\Gamma(R))$ *is strongly connected.*

Proof. Suppose x and y are two non-zero elements of $Z_l(R)$. Then $yx \in Z_l(R)$ gives $x \to 0 \to y$ is a path in $Z_l(\Gamma(R))$. Similarly $xy \in Z_l(R)$, so $y \to 0 \to x$ is a path in $Z_l(\Gamma(R))$. Hence $Z_l(\Gamma(R))$ is strongly connected. □

The above result asserts that $diam(Z_l(\Gamma(R))) = 2$, as the ring R we have considered is non-commutative. In the same way, if $|Z_r(R)| \geq 3$, then $Z_r(\Gamma(R))$ is strongly connected. Also $diam(Z_r(\Gamma(R))) = 2$. However, if R is commutative and $Z(R) > 3$, then diameters of $Z_l(\Gamma(R))$ and $Z_r(\Gamma(R))$ are 1. Observe that in this case, it is immaterial whether $Z_l(R)$ ($Z_r(R)$) is a left (right) ideal or not.

Next we state a lemma which is a very elementary observation.

Lemma 3.1. *Let $f : R_1 \to R_2$ be a ring monomorphism. If $x\ adj\ y$ then $f(x)\ adj\ f(y)$, for $x, y \in R_1$.*

This lemma guides us to the next result in the concept of tournament.

Theorem 3.2. *Let $f : R_1 \to R_2$ be a ring monomorphism. If $T_l(\Gamma(R_1))$ is a tournament, then so is $T_l(\Gamma(f(R_1)))$.*

Proof. Suppose that $T_l(\Gamma(R_1))$ is a tournament. To show $T_l(\Gamma(f(R_1)))$ is also a tournament. For this, we assume $y_1, y_2 \in f(R_1)$. So, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for the elements x_1 and x_2 in R_1 respectively. As $T_l(\Gamma(R_1))$ is a tournament, therefore either $x_1\ adj\ x_2$ or $x_2\ adj\ x_1$. Then from the above lemma we get, either $y_1\ adj\ y_2$ or $y_2\ adj\ y_1$. Thus $T_l(\Gamma(f(R_1)))$ is also a tournament. □

The following lemma provides the inter-relation of ring isomorphism with (left) total directed graph isomorphism and Theorem 5.2 is proved by using the same.

Lemma 3.2. *Let $f : R_1 \to R_2$ be a ring isomorphism. Then f is also an isomorphism from $T_l(\Gamma(R_1))$ onto $T_l(\Gamma(R_2))$.*

Proof. We need only to show that adjacency relation is preserved. For this, we assume that $x\ adj\ y$, for $x, y \in R_1$. Then there exists a non-zero r in $Z_l(R_1)$ with $rx + yr \in Z_l(R_1)$. So

$$\begin{aligned} (rx + yr)z &= 0 \ for\ some\ z(\neq 0) \in R_1 \\ \Rightarrow f((rx + yr)z) &= f(0) \\ \Rightarrow (f(r)f(x) + f(y)f(r))f(z) &= 0. \end{aligned}$$

Since f is an isomorphism, so $f(z) \neq 0$. This gives $f(r)f(x) + f(y)f(r) \in Z_l(R_2)$. Again $f(r) \neq 0$. Thus $f(x)\ adj\ f(y)$. Hence the result. □

There are some other interesting elementary observations noticed in this paper. It enables us to differentiate the very basic concepts left and right. We establish one such result, which relates the left invertible elements of rings with the respective left adjacency relation of left total directed graph.

Theorem 3.3. *Let x, y be two left invertible elements of R. If $x\ adj\ y$ then $x_l^{-1}\ adj\ y_l^{-1}$, where x_l^{-1} and y_l^{-1} are left inverses of x and y respectively.*

Proof. Let $x\ adj\ y$. Then there exists a non-zero r in $Z_l(R)$ with $rx + yr \in Z_l(R)$. So

$$\begin{aligned} (rx + yr)z &= 0 \ for\ some\ z \neq 0 \\ \Rightarrow (y_l^{-1}rx + r)x_l^{-1}xz &= 0 \\ \Rightarrow (rx_l^{-1} + y_l^{-1}r)xz &= 0. \end{aligned}$$

Hence $x_l^{-1}\ adj\ y_l^{-1}$, as $xz \neq 0$. □

Moreover, if x, y are two right invertible elements of R, then $x_r^{-1}\ \overline{adj}\ y_r^{-1}$ whenever $x\ \overline{adj}\ y$, where x_r^{-1} and y_r^{-1} are right inverses of x and y respectively. It is needed to be mentioned here that the ideas of left and right cannot be mixed up.

The following are two results related with sink and source. In [15], we see these two results in the concept of zero-divisor graph under group actions in a non-commutative ring by Han.

Theorem 3.4. *For the rings R and S, if $T_l(\Gamma(R))$ and $T_l(\Gamma(S))$ have no sources (respectively no sinks), then so has $T_l(\Gamma(R \times S))$ (respectively no sinks).*

Proof. Let $(r, s) \in R \times S$ be arbitrary. Then $r \in R$, $s \in S$. So r and s are not sources of $T_l(\Gamma(R))$ and $T_l(\Gamma(S))$ respectively. Thus there is an element $x \in R$ and an element $w \in S$ with $x\ adj\ r$ and $w\ adj\ s$ respectively. If $x\ adj\ r$ then there exists an element $u \in Z_l^*(R)$ such that $ux + ru \in Z_l(R)$. Again, if $w\ adj\ s$ then there exists an element $v \in Z_l^*(S)$ such that $vw+sv \in Z_l(S)$. From this, we get $(u, v) \in Z_l^*(R\times S)$ with $(u, v)(x, w)+(r, s)(u, v) \in Z_l(R\times S)$. Therefore $(x, w)\ adj\ (r, s)$. Hence (r, s) is not a source of $T_l(\Gamma(R \times S))$. □

Corollary 3.1. *For the rings $R_1, R_2, \ldots, R_n$, if each $T_l(\Gamma(R_i))$, $i = 1, 2, 3, \ldots, n$ has no sources (respectively no sinks), then so has $T_l(\Gamma(R_1 \times R_2 \times \cdots \times R_n))$ (respectively no sinks).*

Proof. Let $(r_1, r_2, \ldots, r_n) \in R_1 \times R_2 \times \cdots \times R_n$ be arbitrary. Then $r_i \in R_i, \forall i \in \{1, 2, \ldots, n\}$. So r_i is not a source of $T_l(\Gamma(R_i))$, $\forall i \in \{1, 2, \ldots, n\}$. Thus, there is an element $x_i \in R_i$ with $x_i\ adj\ r_i$, $\forall i \in \{1, 2, \ldots, n\}$. Then there exists an element $u_i \in Z_l^*(R_i)$ such that $u_i x_i + r_i u_i \in Z_l(R_i)$, $\forall i \in \{1, 2, \ldots, n\}$. From this, we get $(u_1, u_2, \ldots, u_n) \in Z_l^*(R_1 \times R_2 \times \cdots \times R_n)$ with $(u_1, u_2, \ldots, u_n)(x_1, x_2, \ldots, x_n) + (r_1, r_2, \ldots, r_n)(u_1, u_2, \ldots, u_n) \in Z_l(R_1 \times R_2 \times \cdots \times R_n)$. Therefore $(x_1, x_2, \ldots, x_n) adj (r_1, r_2, \ldots, r_n)$. Hence $(r_1, r_2, \ldots, r_n)$ is not a source of $T_l(\Gamma(R_1 \times R_2 \times \cdots \times R_n))$. □

Next we observe one more basic result of our discussion.

Theorem 3.5. *Let e_r, e_l be right and left identity elements of R respectively and $Z_l^*(R) \neq \phi$. Then $e_r\ adj\ e_l$.*

Proof. Let $r \in Z_l^*(R)$. Then $re_r = r$ and $e_l r = r$, as e_r and e_l are right and left identity elements of R respectively. Therefore, $re_r + e_l r = 2r \in Z_l(R)$. This gives $e_r\ adj\ e_l$. □

In the same way, if $Z_r^*(R) \neq \phi$, then $e_r\ \overline{adj}\ e_l$.

4. The case when $Z_l(R)$ ($Z_r(R)$) is a left (right) ideal of R

In this section, we consider $Z_l(R)$ ($Z_r(R)$) as a left (right) ideal of R.

Theorem 4.1. *If $|Z_l(R)| > 3$, then $girth(Z_l(\Gamma(R))) = 3$.*

Proof. Since $Z_l(\Gamma(R))$ is strongly connected, so for three non-zero left zero-divisors, we have observed that they are pair-wise adjacent in either direction. Hence $girth(Z_l(\Gamma(R))) = 3$. □

In the same way, if $|Z_r(R)| > 3$, then $girth(Z_r(\Gamma(R))) = 3$.

Theorem 4.2. *For any $x, y \in Reg_l(R)$,* x adj y *if and only if every element of $x + Z_l(R)$ is adjacent to every element of $y + Z_l(R)$.*

Proof. Let $a = x + r_1 \in x + Z_l(R)$, $b = y + r_1 \in y + Z_l(R)$. If $x\ adj\ y$, then there exists a non-zero r in $Z_l(R)$ with $rx + yr \in Z_l(R)$. This gives $r(a - r_1) + (b - r_2)r \in Z_l(R)$ i.e. $(ra + br) - (rr_1 + r_2 r) \in Z_l(R)$. As $Z_l(R)$ is a left ideal of R, so $ra + br \in Z_l(R)$. From this $a\ adj\ b$. Conversely, if $a\ adj\ b$ then there exists a non-zero r in $Z_l(R)$ with $ra + br \in Z_l(R)$. From this $r(x + r_1) + (y + r_2)r \in Z_l(R)$. Therefore $rx + yr \in Z_l(R)$. Hence $x\ adj\ y$. □

In the same way, for any $x, y \in Reg_r(R)$, $x\overline{adj}y$ if and only if every element of $x + Z_r(R)$ is adjacent to every element of $y + Z_r(R)$.

Theorem 4.3. *Let e_l be a left identity element of R. Then $e_l + Z_l(R)$ is a symmetric digraph.*

Proof. Let $e_l + r_1, e_l + r_2$ be any elements of $e_l + Z_l(R)$, where $r_1, r_2 \in Z_l(R)$. Then, at least one of r_1, r_2 is non-zero. From this, we get $(e_l + r_1) adj (e_l + r_2)$, $(e_l + r_2) adj (e_l + r_1)$. Hence $e_l + Z_l(R)$ is a symmetric digraph. □

Also, if e_r is a right identity element, then $e_r + Z_r(R)$ is a symmetric digraph.

Lemma 4.1. *Let $|Z_l(R)| \geq 2$ and e_r be a right identity element. Then, $x + e_r$ is adjacent to every element of $y + Reg_l(R)$, for $x, y \in Z_l(R)$.*

Proof. If $x \neq 0$, then $x(x + e_r) + (y + r_l)x \in Z_l(R)$, and if $y \neq 0$, then $y(x + e_r) + (y + r_l)y \in Z_l(R)$, for any $r_l \in Reg_l(R)$. Hence the result. □

From the above lemma, it is obvious that every element of $x + E_r(R)$ is adjacent to every element of $y + Reg_l(R)$.

Theorem 4.4. *Let $x \in Reg_l(R)$ and $e_r \in E_r(R)$. Then every element of $e_r + Z_l(R)$ is adjacent to every element of $x + Z_l(R)$.*

Proof. Let $Z_l^*(R) \neq \phi$, $x \in Reg_l(R)$ and $e_r \in E_r(R)$. Then $e_r\ adj\ x$. □

Theorem 4.5. *Let $Z_l^*(R) \neq \phi$, $x \in Reg_l(R)$ and $e_r \in E_r(R)$. Then e_r* adj x.

Proof. Let $z \in Z_l^*(R)$. Then $ze_r + xz \in Z_l(R)$. Hence $e_r\ adj\ x$. □

The above results imply that if $Z_l^*(R) \neq \phi$, then every right identity element is adjacent to every left regular element of R. In the same way, if $Z_r^*(R) \neq \phi$, then every left identity element is adjacent to every right regular element of R. Again, we observe that the graphs with $E_r(R) + Z_l(R)$ and $E_l(R) + Z_r(R)$ as vertex sets respectively, are two symmetric digraphs. Every element of $E_r(R) + Z_l(R)$ is adjacent to every element of $Reg_l(R) + Z_l(R)$. Similarly, every element of $E_l(R) + Z_r(R)$ is adjacent to every element of $Reg_r(R) + Z_r(R)$.

Theorem 4.6. *If $|Z_l(R)| \geq 2$, then $Z_l(R)$ is not disjoint from $Reg_l(R)$.*

Proof. Let $z \in Z_l^*(R)$. Then for $z'(\neq z) \in Z_l(R)$ and $r \in Reg_l(R)$, we get $zz' + rz \in Z_l(R)$. Hence $Z_l(R)$ is not disjoint from $Reg_l(R)$. □

Corollary 4.1. *If $|Z_l(R)| \geq 2$, then $Z_l(R)$ is not disjoint from $Reg_l(R) + Z_l(R)$.*

The dual concept also holds for the immediate preceding theorem and corollary respectively. As a consequence, we obtain that $Reg_l(R)$ and $Reg_r(R)$ are not totally disconnected if and only if $Reg_l(R) + Z_l(R)$ and $Reg_r(R) + Z_r(R)$ are not totally disconnected, respectively.

5. Coloring of total directed graphs

In this section, we discuss some results of coloring of (left) total directed graphs. Actually, we develop this coloring idea of total directed graph with the help of concept of clique. We consider mostly left total directed graph. Exact results can be obtained for coloring of right total directed graph, whenever necessary. Theorem 4.5 is an interesting part of our discussion as it directly relates $T_l(\Gamma(R))$ with $T_r(\Gamma(R))$.

Theorem 5.1. *Let R be a reduced ring. Then $\chi(T_l(\Gamma(R))) = 1$ if and only if R has no non-zero left zero-divisors.*

Proof. Suppose x is a non-zero left zero divisor of R. Then there exists a non-zero y in R with $xy = 0$. If $x = y$, then $x^2 = 0$. But R is a reduced ring, so $x = 0$, a contradiction. Also, if $x = x + y$, then $y = 0$, again a contradiction. This gives x is distinct from y and $x + y$, so $xy + (x + y)x \in Z_l(R)$. Thus $y\ adj\ (x + y)$. But it is a contradiction to $\chi(T_l(\Gamma(R))) = 1$. Hence R has no non-zero left zero-divisors. □

Theorem 5.2. *Suppose R has no left identity and $Z_l(R)$ is a not an ideal of R. If there exists a non-zero element x in R with $x^2 = 0$ and $|Rx| = n$, then $\chi(T_l(\Gamma(R/Z_l(x)))) \geq n$.*

Proof. Let $r_1x, r_2x \in Rx$. As $x(r_1x) + (r_2x)x = (xr_1)x \in Z_l(R)$. So $r_1x\ adj\ r_2x$. From this, we get Rx is a clique. Now $Rx \cong R/Z_l(x)$. Therefore, $T_l(\Gamma(Rx)) \cong T_l(\Gamma(R/Z_l(x)))$, by Lemma 3.2. This gives $R/Z_l(x)$ is also a clique, since Rx is a clique. Thus $\chi(T_l(\Gamma(R/Z_l(x)))) \geq n$. Hence the result. □

Theorem 5.3. *If $y_jy_i = y_ky_i$ for $y_i, y_j, y_k \in R$ and $k > j > i$, then $T_l(\Gamma(R))$ contains an infinite clique.*

Proof. Let $y_jy_i = y_ky_i$ for $y_i, y_j, y_k \in R$ and $k > j > i$. If $z_{i,j} = y_i - y_j$, $j > i$, then

$$\begin{aligned} z_{k,r}z_{i,j} &= (y_k - y_r)(y_i - y_j) \\ &= y_ky_i - y_ky_j - y_ry_i + y_ry_j \\ &= 0. \end{aligned}$$

Thus $z_{k,r} \in Z_l^*(R)$ and so $\{z_{3,4}, z_{3,5}, \ldots\} \subseteq Z_l^*(R)$. Now $z_{3,4}z_{1,2} + z_{3,5}z_{3,4} \in Z_l(R)$. This gives $\{z_{1,2}, z_{3,5}\}$ is a clique. If $z_{6,7} \notin \{z_{1,2}, z_{3,5}\}$, then $z_{3,4}z_{1,2} + z_{6,7}z_{3,4} \in Z_l(R)$ and $z_{8,9}z_{3,5} + z_{6,7}z_{8,9} \in Z_l(R)$. From this, $\{z_{1,2}, z_{3,5}, z_{6,7}\}$ is a clique. Proceeding in this way, we get an infinite clique in $T_l(\Gamma(R))$. Hence the result. □

Theorem 5.4. *If I is a finite ideal in R, then $T_l(\Gamma(R))$ contains an infinite clique if and only if $T_l(\Gamma(\frac{R}{I}))$ has an infinite clique.*

Proof. If R has an infinite clique C, the homomorphic image $\overline{C}$ of C is a clique in $T_l(\Gamma(\overline{R}))$ where $\overline{R} = R/I$, and since I is finite, $\overline{C}$ is still infinite. We assume C is an infinite clique. If $x \ adj \ y$, for $x, y \in C$, then there exists a non-zero $r \in Z_l(R)$ distinct from x and y with $rx + yr \in Z_l(R)$. This gives $(r+I)(x+I)+(y+I)(r+I) \in Z_l(R/I)$. Thus $(x + I) \ adj \ (y + I)$. Therefore $\overline{C}$ is a clique. Conversely, let $\{\overline{x_i}\}$ be a clique in $\overline{C}$. Then it is easy to verify that $\{x_i\}$ is a clique in C. Hence the theorem. □

Theorem 5.5. *Let x be a nilpotent element of degree $n \geq 3$ of $T_l(\Gamma(R))$, which does not contain any right identity element. If every right ideal of R is a left ideal and $x^{n-1} \notin x^2R$, then there is an infinite clique in $T_r(\Gamma(R))$.*

Proof. Let $x^n = 0$, $n \geq 3$. If we put $y = x^2$, then $y^{n-1} = (x^2)^{n-1} = 0$. If yR is infinite, then $T_l(\Gamma(R))$ has an infinite clique. Let yR be finite. This gives xR/yR is infinite. Now xR/yR is an infinite clique in $\overline{R} = R/yR$. As

$$\begin{aligned} \overline{r} &= x^{n-1} + yR \\ &\neq yR. \end{aligned}$$

and

$$\begin{aligned} \overline{r}^2 &= (x^{n-1} + yR)(x^{n-1} + yR) \\ &= x^{2n}x^{-2} + yR \\ &= yR. \end{aligned}$$

Let $\overline{r_1}, \overline{r_2} \in xR/yR$. Then $\overline{r_1} = r_1x + yR$, $\overline{r_2} = r_2x + yR$. Since

$$\begin{aligned} \overline{r}\,\overline{r_1} + \overline{r_2}\,\overline{r} &= (x^{n-1} + yR)(r_1x + yR) + (r_2x + yR)(x^{n-1} + yR) \\ &= (x^{n-1} + yR)(r_1x + yR) + r_2x^n + yR \\ &= (x^{n-1} + yR)(r_1x + yR) \\ &\in Z_r(R/yR). \end{aligned}$$

Therefore, $\overline{xR} = xR/yR$ is an infinite clique in $T_r(\Gamma(\overline{R}))$ where $\overline{R} = R/yR$. As yR is finite, so $T_r(\Gamma(R))$ has an infinite clique. Hence the result. □

Conclusion : In this paper, we have defined total directed graph of non-commutative ring and have discussed some basic results. We have also investigated some properties of coloring of total directed graphs. In fact, the coloring idea of this article has been observed from the most motivating paper of graphical aspects of algebraic structures entitled 'Coloring of commutative rings [16]', introduced by Istvan Beck in 1988.

References

[1] D.F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008) 2706–2719.

[2] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, The total graph and regular graph of a commutative ring, J. Pure Appl. Algebra 213 (2009) 2224–2228.

[3] A. Abbasi, S. Habibi, On the total graph of a commutative ring with respect to proper ideals, J. Korean Math. Soc. 49 (1) (2012) 85–98.

[4] D.F. Anderson, A. Badawi, The total graph of a commutative ring without the zero element, J. Algebra Appl. 11 (4) (2012) 1250074.

[5] M.H. Shekarriz, M.H.S. Haghighi, H. Sharif, On the total graph of a finite commutative ring, Comm. Algebra 40 (2012) 2798–2807.

[6] A. Abbasi, S. Habibi, The total graph a module over a commutative ring with respect to proper submodules, J. Algebra Appl. 11 (3) (2012) 1250048.

[7] S.E. Atani, S. Habibi, The total torsion element graph of a module over a commutative ring, An. St. Univ. Ovidius Constant 19 (1) (2011) 23–34.

[8] M.F. Atiyah, I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley, London, 1969.

[9] R. Balkrishnan, K. Ranganathan, A Text Book of Graph Theory, Springer-verlag, New York, Inc., 2008. Reprint.
[10] F. Harary, Graph theory, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1969.
[11] J.A. Huckaba, Commutative Rings with Zero-Divisors, Marcel-Dekker, New York, Basel, 1988.
[12] J. Lambeck, Lectures on Rings and Modules, Blaisdell Publishing Company, Waltham, Toronto, London, 1966.
[13] I. Kaplansky, Commutative Rings, revised Edition, University of Chicago Press, Chicago, 1974.
[14] T. Wu, On directed zero-divisor graphs of finite rings, Discrete Math. 296 (1) (2005) 73–86.
[15] J. Han, The zero-divisor graph under group actions in a non-commutative ring, J. Korean Math. Soc. 45 (2008) 1647–1659.
[16] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988) 208–226.